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Model Uncertainty in Finance and Second Order Backward Stochastic Differential Equations

Incertitude sur les Modèles en Finance et Équations Différentielles Stochastiques Rétrogrades du Second Ordre

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Résumé

L'objectif principal de cette thèse est d'étudier quelques problèmes de mathématiques financières dans un marché incomplet avec incertitude sur les modèles. Récemment, deux approches différentes (mais liées) ont été développées sur ce sujet. L'une est la théorie des G-espérances non-linéaires initiée par Peng [89], et l'autre est la théorie des équations différentielles stochastiques rétrogrades du second ordre (dans la suite 2EDSRs) introduite par Soner, Touzi et Zhang [101]. Dans cette thèse, nous adoptons le point de vue de ces derniers auteurs.

Cette thèse contient quatre parties dans le domain des 2EDSRs. Nous commençons par généraliser la théorie des 2EDSRs initialement introduite dans le cas de générateurs lipschitziens continus à celui de générateurs à croissance quadratique. Cette nouvelle classe des 2EDSRs nous permettra ensuite d'étudier le problème de maximisation d'utilité robuste dans les modèles non-dominés, ce qui peut être considéré comme une extension non-linéaire du problème de maximisation d'utilité standard. Dans la deuxième partie, nous étudions ce problème pour les fonctions d'utilité exponentielle, puissance et logarithmique. Dans chaque cas, nous donnons une caractérisation de la fonction valeur et d'une stratégie d'investissement optimale via la solution d'une 2EDSR.

Dans la troisième partie, nous fournissons également une théorie d'existence et unicité pour des EDSRs réfléchies du second ordre avec obstacles inférieurs et générateurs lipschitziens, nous appliquons ensuite ce résultat à l'étude du problème de valorisation des options américaines dans un modèle financier à volatilité incertaine. Dans la quatrième partie, nous étudions une classe des 2EDSRs avec sauts. En particulier, nous prouvons l'existence et l'unicité de solutions dans les espaces appropriés. Nous pouvons interpréter ces équations comme des EDSRs standards avec sauts, avec volatilité et mesure de saut incertaines. Ces équations sont les candidats naturels pour l'interprétation probabiliste des équations aux dérivées partielles intégro-différentielles complètement non-linéaires. Comme application de ces résultats, nous étudions un problème de maximisation d'utilité exponentielle robuste avec incertitude sur les modèles. L'incertitude affecte à la fois le processus de volatilité, mais également la mesure des sauts.

La dernière partie est dédiée à l'implémentation numérique des méthodes de Monte Carlo pour la valorisation des options dans des modèles à volatilité incertaine. Ce travail pratique a été réalisé lors d'un stage au cours de la première année de thèse.

Mots-clés: Équations diffèntielles stochastiques rétrogrades du second ordre, mesures de probabilités mutuellement singulières, analyse stochastique quasi-sûre, formule de Feynman-Kac non-linéaire, EDPs complètement non-linéaires, générateur à croissance quadratique, maximisation d'utilité robuste, incertitude sur les modèles, problème d'obstacle, options américaines, temps d'arrêt optimal,équations différentielles stochastiques rétrogrades avec sauts.

Abstract

The main objective of this PhD thesis is to study some financial mathematics problems in an incomplete market with model uncertainty. In recent years, two different, but somewhat linked, frameworks have been developed on this topic. One is the nonlinear G-expectation introduced by Peng [89], and the other one is the theory of second order backward stochastic differential equations (2BSDEs for short) introduced by Soner, Touzi and Zhang [101]. In this thesis, we adopt the latter point of view.

This thesis contains of four key parts related to 2BSDEs. In the first part, we generalize the 2BSDEs theory initially introduced in the case of Lipschitz continuous generators to quadratic growth generators. This new class of 2BSDEs will then allow us to consider the robust utility maximization problem in non-dominated models, which can be regarded as a nonlinear extension of the standard utility maximization problem. In the second part, we study this problem for exponential utility, power utility and logarithmic utility. In each case, we give a characterization of the value function and an optimal investment strategy via the solution to a 2BSDE.

In the third part, we provide an existence and uniqueness result for second order reflected BSDEs with lower obstacles and Lipschitz generators, and then we apply this result to study the problem of American contingent claims pricing with uncertain volatility. In the fourth part, we define a notion of 2BSDEs with jumps, for which we prove the existence and uniqueness of solutions in appropriate spaces. We can interpret these equations as standard BSDEs with jumps, under both volatility and jump measure uncertainty. These equations are the natural candidates for the probabilistic interpretation of fully nonlinear partial integro-differential equations. As an application of these results, we shall study a robust exponential utility maximization problem under model uncertainty, where the uncertainty affects both the volatility process and the jump measure.

The last part is about numerical implementation of Monte Carlo schemes for options pricing in uncertain volatility models, which was realized during an internship during the first year of this PhD study.

Keywords: Second order backward stochastic differential equations, mutually singular probability measures, quasi-sure stochastic analysis, fully nonlinear PDEs, nonlinear Feynman-Kac formula, quadratic growth generator, robust utility maximization, model uncertainty, obstacle problem, American contingent claims, optimal stopping time, backward stochastic differential equations with jumps.

Sommaire

Chapitre 1

Introduction

The main objective of this PhD thesis is to study some financial mathematics problems in an incomplete market with model uncertainty. In recent years, two different, but somewhat linked, frameworks have been developed on this topic. One is the nonlinear G-expectation introduced by Peng [89], and the other one is the theory of second order backward stochastic differential equations (2BSDEs for short) introduced by Soner, Touzi and Zhang [101]. In this thesis, we adopt the latter point of view.

This thesis contains four key chapters related to 2BSDEs. We first generalize the 2BSDEs theory initially introduced in the case of Lipschitz continuous generators to quadratic growth generators in Chapter 2. This new class of 2BSDEs will then allow us to study the robust utility maximization problem in non-dominated models, which can be regarded as a nonlinear extension of the standard utility maximization problem. In Chapter 3, we study this problem for exponential utility, power utility and logarithmic utility. In each case, we give a characterization of the value function and an optimal investment strategy via the solution to a 2BSDE. In Chapter 4, we also provide an existence and uniqueness theoty for second order reflected BSDEs (2RBSDEs for short) with one lower obstacle and Lipschitz generators, then apply this result to study the problem of American contingent claims pricing with uncertain volatility.

In Chapter 5, we define a notion of 2BSDEs with jumps, for which we prove the existence and uniqueness of solutions in appropriate spaces. We can interpret these equations as standard BSDEs with jumps, under both volatility and jump measure uncertainty. These equations are the natural candidates for the probabilistic interpretation of fully nonlinear partial integro-differential equations. As an application of these results, we shall study a robust exponential utility maximization problem under model uncertainty. The uncertainty affects both the volatility process and the jump measure.

The last chapter (6) is about numerical implementation of Monte Carlo schemes for options pricing with uncertain volatility models, which I realized during an internship at Crédit Agricole CIB during the first year of my PhD study.

Backward stochastic differential equations (BSDEs for short) first appeared in Bismut [11] in the linear case, and then have been widely studied since the seminal paper of Pardoux and Peng [87]. Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}, \mathbb{P})$ generated by an $\mathbb{R}^d$-valued Brownian motion W, a solution to a BSDE consists of a pair of progressively measurable processes (Y, Z) such that

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ t \in [0,T],\ \mathbb{P} - a.s. \tag{1.0.1}$$

where f (called the generator) is a progressively measurable function and ξ (called the terminal condition) is an $\mathcal{F}_T$-measurable random variable. Pardoux and Peng proved existence and uniqueness of the above BSDE provided that the function f is uniformly Lipschitz in y and z and that ξ and $f_s(0,0)$ are square integrable. In the particular case when the randomness in f and ξ is induced by the current value of a state process defined by a forward stochastic differential equation, the solution to the so called Markovian BSDE could be linked to the solution of a semilinear PDE by means of a generalized Feynman-Kac formula. Since their pioneering work, many efforts have been made to relax the assumptions on the generator f; for instance, Lepeltier and San Martin [67] have proved the existence of a solution when f is only continuous in (y,z) with linear growth. Most of these efforts are particularly motivated by applications of BSDEs in many fields such as: financial mathematics, stochastic games, semilinear PDEs, stochastic controls, etc. We refer to El Karoui, Peng and Quenez [33] for a review of these applications.

The link between BSDEs and semilinear PDEs is important for the formulation of 2BSDEs. Therefore let us show it with the following example. Consider the parabolic PDE:

$$\begin{cases} (\partial_t + \mathcal{L})u(t,x) + f(t,x,u(t,x),\sigma^* Du(t,x)) = 0 \\ u(T,x) = g(x) \end{cases} \tag{1.0.2}$$

where $\mathcal{L}$ is the second order differential operator defined as follows

$$\mathcal{L}\varphi(x) := \sum_{i=1}^{d} b^i(x)\partial_{x_i}\varphi(x) + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)^{ij}(x)\partial^2_{x_i x_j}\varphi(x) \quad \varphi \in C^2(\mathbb{R}^d).$$

If g, f and the coefficients of the operator $\mathcal{L}$ are smooth enough, the PDE(1.0.2) has a classic solution $u \in C^{1,2}$. Then the processes $(Y,Z) = (Y_s^{t,x}, Z_s^{t,x}) := (u(s,X_s^{t,x}), \sigma^* Du(s,X_s^{t,x}))$ solves the following BSDE:

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r,$$

where $(X_s^{t,x})_{t \leqslant s \leqslant T}$ is the diffusion process associated with the operator $\mathcal{L}$ starting from x at t. In particular, $u(t,x) = Y_t^{t,x}$, and $\sigma^* Du(t,x) = Z_t^{t,x}$ which is a generalization of the well known Feynman-Kac formula to a semilinear case.

More recently, motivated by applications in financial mathematics and probabilistic numerical methods for PDEs (see [20], [41], [91] and [100]), Cheridito, Soner, Touzi and Victoir [22] introduced the first formulation of second order BSDEs, which are connected to the larger class of fully nonlinear PDEs. Then, Soner, Touzi and Zhang [101] provided a new formulation of 2BSDEs based on quasi-sure stochastic analysis. Their key idea was to consider a family of BSDEs defined quasi surely (q.s. for short) under a non-dominated class of mutually singular probability measures,which means $\mathbb{P}-a.s.$ for every probability measure $\mathbb{P}$ in this class.

We first give some intuition in one dimensional case which will help to well understand the new formulation of 2BSDEs. Let $H_t(y,z,\gamma) := G(\gamma) := \frac{1}{2}\sup_{\underline{a} \leqslant a \leqslant \overline{a}}(a\gamma) = \frac{1}{2}\left(\overline{a}\gamma^+ - \underline{a}\gamma^-\right)$ with $0 < \underline{a} \leqslant \overline{a} < \infty$, and suppose that the following fully nonlinear PDE

$$\begin{cases} \partial_t u + G(D^2 u) = 0 \\ u(T,.) = \Phi \end{cases}$$

has a smooth solution. The process $X_t^\alpha := \int_0^t \alpha_r^{1/2} dW_r$ is well defined with $(\alpha_r)_{0 \leqslant r \leqslant T}$ a process taking values in $[\underline{a}, \overline{a}]$. Then the pair $(Y_t := u(t, X_t^\alpha),\ Z_t := Du(t, X_t^\alpha))$ satisfies the following equation

$$Y_t = \Phi(X_T^\alpha) - \int_t^T Z_s dX_s^\alpha + K_T - K_t$$

with $K_t := \int_0^t \left(G(D^2 u) - \frac{1}{2}\alpha_s D^2 u \right)(s, X_s^\alpha) ds$. In particular, we notice that K is a nondecreasing process such that $K_0 = 0$. Thus, it is natural that there is some nondecreasing process appearing in the formulation of 2BSDEs.

Next, with a similar example, we suggest a representation for the solution Y of 2BSDEs. Let u be a solution of the following fully nonlinear PDE

$$\partial_t u + H(., u, Du, D^2 u) = 0 \text{ and } u(T, .) = \Phi$$

with $H(t, x, r, p, \gamma) = \sup_{a > 0} \left\{ \frac{1}{2} a\gamma - f(t, x, r, p, a) \right\}$. Then we should have, formally, $u = \sup\limits_{a \in D_f} u^a$ where D_f denote the definition domain of f in a on $\mathbb{R}_+^*$ and u^a is a solution of

$$\partial_t u^a + \frac{1}{2} a D^2 u^a - f(., u^a, Du^a, a) = 0 \text{ and } u^a(T, .) = \Phi.$$

Since the above PDE is semilinear, it corresponds to a BSDE. This provides a possible candidate for the solution Y to the Markovian 2BSDE associated to the fully nonlinear PDE. We should have, again formally, $Y_t = \sup\limits_\alpha Y_t^\alpha$ with

$$Y_s^\alpha = \Phi(X_T^\alpha) - \int_s^T f(r, X_r^\alpha, Y_r^\alpha, Z_r^\alpha, \alpha_r) dr - \int_s^T Z_r^\alpha \alpha_r^{1/2} dW_r,\ s \in [t, T],$$

where $(\alpha_r)_{t \leqslant r \leqslant T}$ is a positive process taking values in D_f and where $X_s^\alpha = x + \int_t^s \alpha_r^{1/2} dW_r$.

With the above examples in mind, we will now give a rigorous description of this framework. Let $\Omega := \left\{ \omega \in C([0, T], \mathbb{R}^d) : \omega_0 = 0 \right\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0 \leqslant t \leqslant T} |\omega_t|$, B the canonical process.

We define F as the corresponding conjugate of a given map H w.r.t.γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} \mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}_d^{>0},$$

where $\mathbb{S}_d^{>0}$ denotes the set of all real valued positive definite $d \times d$ matrices. And

$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t)$$

with $\widehat{a}_t := \limsup\limits_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left(\langle B \rangle_t - \langle B \rangle_{t-\varepsilon} \right)$, where $\langle B \rangle_t := B_t B_t^T - 2\int_0^t B_s dB_s^T$ is defined pathwise and the $\limsup$ is taken componentwise.

We denote by $\mathcal{P}_H$ the non-dominated class of mutually singular probability measures, where under each $\mathbb{P} \in \mathcal{P}_H$, $\widehat{a}$ has positive finite bounds which may depend on $\mathbb{P}$. We shall consider the following 2BSDE,

$$Y_t = \xi - \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \leqslant t \leqslant T,\ \mathcal{P}_H - q.s.. \tag{1.0.3}$$

Definition 1.0.1. *We say (Y, Z) is a solution to 2BSDE (1.0.3) if :*

- $Y_T = \xi$, $\mathcal{P}_H - q.s.$
- *For all $\mathbb{P} \in \mathcal{P}_H$, the process $K^{\mathbb{P}}$ defined below has nondecreasing paths $\mathbb{P} - a.s.$*

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \leqslant t \leqslant T,\ \mathbb{P} - a.s. \tag{1.0.4}$$

- *The family $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\right\}$ satisfies the minimum condition*

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\right],\ 0 \leqslant t \leqslant T,\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H. \tag{1.0.5}$$

  where $\mathcal{P}_H(t^+, \mathbb{P})$ is the set of probability measures in $\mathcal{P}_H$ which coincide with $\mathbb{P}$ until t^+.

Moreover if the family $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\right\}$ can be aggregated into a universal process K, we call (Y, Z, K) a solution of 2BSDE (1.0.3).

The above minimum condition can be understood as that K is a martingale under the nonlinear expectation generated by the set of probability measures $\mathcal{P}_H$.

Under uniform Lipschitz conditions similar to those of Pardoux and Peng, Soner, Touzi and Zhang [101] established a complete theory of existence and uniqueness for the solution to the above 2BSDE. Possamaï in [90] extended their results to the case of a continuous linear growth generator. In the following, we will concentrate ourselves on this new formulation.

1.1 Second Order BSDEs with Quadratic Growth Generators

Motivated by a robust utility maximization problem under volatility uncertainty, in this part of the thesis, we generalize the 2BSDEs theory to the case where the generators have quadratic growth in z.

Quadratic BSDEs in the classical case was first studied by Kobylanski [63], who proved existence and uniqueness of a solution by means of approximation techniques borrowed from the PDE literature, when the generator is continuous and has quadratic growth in z and the terminal condition ξ is bounded. Tevzadze in [107] has given a direct proof for the existence and uniqueness of a bounded solution in the Lipschitz-quadratic case, proving the convergence of the usual Picard iteration. Recently, Briand and Hu [12] have extended the existence result to unbounded terminal condition with exponential moments and proved uniqueness for a convex coefficient [13]. Finally, Barrieu and El Karoui [6] recently adopted a completely different approach, embracing a forward point of view to prove existence under conditions similar to those of Briand and Hu. Quadratic BSDEs find their applications essentially in dynamic risk measures and utility maximization under constraints.

For 2BSDEs with quadratic growth generators, our main assumptions on the function F is as follows

Assumption 1.1.1. (i) *$\mathcal{P}_H$ is not empty, and the domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *F is $\mathbb{F}$-progressively measurable in D_{F_t}.*

(iii) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

(iv) *F is continuous in z and has the following growth property. There exists $(\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$ such that*

$$\left|\widehat{F}_t(y,z)\right| \leqslant \alpha + \beta\,|y| + \frac{\gamma}{2}\left|\widehat{a}^{1/2}z\right|^2, \mathcal{P}_H - q.s., \text{ for all } (t,y,z).$$

(v) *F is C^1 in y and C^2 in z, and there are constants r and θ such that for all (t,y,z),*

$$|D_y\widehat{F}_t(y,z)| \leqslant r, \ \ |D_z\widehat{F}_t(y,z)| \leqslant r + \theta\left|\widehat{a}^{1/2}z\right|,$$

$$|D^2_{zz}\widehat{F}_t(y,z)| \leqslant \theta, \ \ \mathcal{P}_H - q.s..$$

Among the above assumptions, (i) and (iii) are taken from [101] and are needed to deal with the technicalities induced by the quasi-sure framework; (ii) and (iv) are quite standard in the classical BSDEs literature; and (v) introduced in Tevzadze [107] is essential to prove existence of a solution to quadratic 2BSDEs.

The main difference with the case of Lipschitz generators is the quadratic growth assumptions on z, which induce many technical difficulties in our framework. As for the BSDEs with quadratic growth, we show that the Z-part of a solution to 2BSDEs also satisfies certain BMO property. This property plays a very important role in the proof for 2BSDEs, much more than for the classical BSDEs.

With a generalization of the comparison theorem proved in [107] (see Theorem 2), we then obtain a representation formula for solution to 2BSDE as in Theorem 4.4 of [101].

Theorem 1.1.1. *Let Assumptions 1.1.1 hold. Assuming that $\xi \in \mathbb{L}^\infty_H$ and $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ (the solution space, see Chapter 2 for precise definition) is a solution to 2BSDE (1.0.3). Then, for any $\mathbb{P} \in \mathcal{P}_H$ and $0 \leqslant t_1 < t_2 \leqslant T$,*

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})} y^{\mathbb{P}'}_{t_1}(t_2, Y_{t_2}), \ \ \mathbb{P} - a.s. \tag{1.1.1}$$

where $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau,\xi), z^{\mathbb{P}}(\tau,\xi))$ is the unique solution of the classical BSDE with the same generator $\widehat{F}$ (existence and uniqueness have been proved under our assumptions *by Tevzadze in [107]), for any $\mathbb{P} \in \mathcal{P}_H$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^\infty(\mathbb{P})$.*

Consequently, the 2BSDE (1.0.3) has at most one solution in $\mathbb{D}^\infty_H \times \mathbb{H}^2_H$.

To prove existence of a solution, we generalize the approach in the article [101] to the quadratic case, where the main tool is the so-called regular conditional probability distributions of Stroock and Varadhan [104]. This allows to construct a solution to the 2BSDE when the terminal condition belongs to the space $\mathrm{UC_b}(\Omega)$. Then, by passing to limit, we prove existence of solution when the terminal condition is in $\mathcal{L}^\infty_H$, the closure of $\mathrm{UC_b}(\Omega)$ under a certain norm defined in Chapter 2.

Theorem 1.1.2. *Let $\xi \in \mathcal{L}^\infty_H$. Under Assumption 1.1.1, there exists a unique solution $(Y, Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ to the 2BSDE (1.0.3).*

Indeed, this approach relies very heavily on the Lipschitz and Lipschitz-quadratic assumption on the generator. Besides, it can only be used if we are able first to prove uniqueness of the solution through a representation property. This is why we put some efforts to provide another proof of existence based on approximation techniques similar to those used in the classical BSDEs literature recalled above. But, since we are working under a family of mutually singular probability measures which is not necessarily weakly compact, both the classical monotone convergence theorem and the one proved by Denis, Hu and Peng [28] in the framework of G-expectation can not be applied in our framework. So the second approach will be left for future research.

Finally, we consider Markovian 2BSDEs with quadratic growth generators, whose solution can be represented by a deterministic function of t and B_t, and show the connection of these 2BSDEs with fully nonlinear PDEs.

We define f and $\widehat{h}$ as the corresponding conjugate and bi-conjugate functions of a deterministic map h. Our object of interest is the following Markovian 2BSDE with terminal condition $\xi = g(B_T)$

$$Y_t = g(B_T) - \int_t^T f(s, B_s, Y_s, Z_s, \widehat{a}_s)ds - \int_t^T Z_s dB_s + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ \mathcal{P}_h - q.s.$$

We establish the connection $Y_t = v(t, B_t)$, $\mathcal{P}_h - q.s.$, where v is the solution in some sense of the following fully nonlinear PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) + \widehat{h}\,(t, x, v(t,x), Dv(t,x), D^2 v(t,x)) = 0,\ t \in [0,T) \\ v(T,x) = g(x). \end{cases} \tag{1.1.2}$$

1.2 Robust Utility Maximization in Non-dominated Models

After establishing the result of uniqueness and existence of solution to 2BSDE with quadratic growth generators, we are ready to study the robust utility maximization problem. The problem of utility maximization, in its most general form, can be formulated as follows

$$V^{\xi}(x) \ := \ \sup_{\pi\in\mathcal{A}}\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}[U(X_T^{\pi}-\xi)],$$

where $\mathcal{A}$ is a given set of admissible trading strategies, $\mathcal{P}$ is the set of all possible models, U is a utility function, X_T^{π} is the liquidation value of a trading strategy π with positive initial capital $X_0^{\pi}=x$ and ξ is a terminal liability, equal to 0 if U is only defined on $\mathbb{R}^+$.

In the standard problem of utility maximization, $\mathcal{P}$ contains only one probability measure $\mathbb{P}$. This means that the investor knows the "historical" probability $\mathbb{P}$ that describes the dynamics of the underlying asset. But, in reality, the investor may have some uncertainty on this probability, which means that there can be several objective probability measures in $\mathcal{P}$. In this case, we call the problem robust utility maximization. Many authors introduce a dominated set of probability measures which are absolutely continuous with respect to a reference probability measure $\mathbb{P}$. This is going to be the case if we only take into account drift uncertainty. However, if we want to work in the framework of uncertain volatility models (UVM for short) introduced by Avellaneda, Lévy and Paras. [2] and Lyons [75], the set of probability measures becomes non-dominated.

After the pioneer work of Von Neumann and Morgenstern [109], Merton first studied portfolio selection with utility maximization by stochastic optimal control in the seminal paper [81]. Kramkov and Schachermayer solved the problem of maximizing utility of final wealth in a general semimartingale model by means of duality in [64]. Later, El Karoui and Rouge [38] considered the indifference pricing problem via exponential utility maximization by means of the BSDE theory. Their strategy set is supposed to be closed and convex, and the problem is solved using BSDEs with quadratic growth generators. In [54], with a similar approach, Hu, Imkeller and Müller studied three important types of utility function with only closed admissible strategies set within incomplete market and found that the maximization problem is linked to quadratic BSDEs. They also showed a deep link between quadratic growth and the BMO spaces. Morlais [82] extended results in [54] to more general continuous filtration, for this purpose, proved existence and uniqueness of the solution to a particular type quadratic BSDEs driven by a continuous martingale. In a more recent paper [57], Jeanblanc, Matoussi and Ngoupeyou studied the indifference price of an unbounded claim in an incomplete jump-diffusion model by considering the risk aversion represented by an exponential utility function. Using the dynamic programming equation, they found the price of an unbounded credit derivatives as a solution of a quadratic BSDE with jumps.

The problem of robust utility maximization with dominated models was introduced by Gilboa and Schmeidler [44]. An example of this case is when the drift is uncertain. Anderson, Hansen and Sargent [1] and Hansen et al. [53] then introduced and discussed the basic problem of robust utility maximization penalized by a relative entropy term of the model uncertainty $\mathbb{Q}\in\mathcal{P}$ with respect to a given reference probability measure $\mathbb{P}_0$. Inspired by these latter works, Bordigoni, Matoussi and Schweizer [15] considered the robust problem in a general context of semimartingale by stochastic control and proved that the solution of this problem is a solution of a particular BSDE. In Müller's thesis

[84], he studied the robust problem in the case when the drift is unknown with BSDEs theory. Some results in the robust maximization problem have also been obtained with convex duality. We can refer to Gundel [46] , Quenez [94], Schied [97], Schied and Wu [98], Skiadas [99] in the case of continuous filtration, among others,

To our best knowledge, robust utility maximization with non-dominated models, encompassing the case of the UVM framework, was first studied with duality theory by Denis and Kervarec [29]. In the article, they took into account uncertainty about both the volatility and the drift. The utility function U in their framework was supposed to be bounded and to satisfy some conditions as in the classical case. They first established a dual representation for robust utility maximization and then they showed that there exists a least favorable probability which means that solving the robust problem is equivalent to solving the standard problem under this probability. More recently, Tevzadze et al. [108] studied a similar robust utility maximization problem for exponential and power utility functions (and also for mean-square error criteria), by means of the dynamic programming approach already used in [105]. They managed to show that the value function of their problem solves a PDE. We will compare their results with ours in Section 3.7 of Chapter 3.

In our framework, we study robust utility maximization with non-dominated models, more precisely UVM where $\widehat{a}$ has uniform positive finite bounds, via 2BSDEs theory. Meanwhile, our set of mutually singular probability measures is more restrictive than in [29]. We study the problem for exponential utility, power utility and logarithmic utility, which, unlike in [29], are not bounded. In particular, we prove the existence of optimal strategy and provide characterization of value function via solution to 2BSDEs. Moreover, for exponential utility, the result also gives us the indifference price for a contingent claim payed at a terminal date in the case of UVM. Then it allows us to price and hedge contingent claim in a market where some external risks can't be hedged. At the end, we also give some examples where we can explicitly solve the robust utility maximization problems by finding the solution to the associated 2BSDEs, and we try to give some intuitions and comparisons with the classical framework of Merton's PDEs.

To find the value function $V^{\xi}(x)$ and an optimal trading strategy π^*, we follow the main ideas of the general *martingale optimality principle* approach as in [38] and [54], but adapting it here to a non-dominated models framework.

Let $\mathcal{A}$ be the set of admissible trading strategies. We construct R^{π} a family of processes which satisfies the following properties:

Properties 1.2.1. (i) $R^{\pi}_T = U(X^{\pi}_T - \xi)$ *for all* $\pi \in \mathcal{A}$

(ii) $R^{\pi}_0 = R_0$ *is constant for all* $\pi \in \mathcal{A}$

(iii) *We have*

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[U(X^{\pi}_T - \xi)] \leqslant R^{\pi}_t, \ \forall \pi \in \mathcal{A}$$

$$R^{\pi^*}_t = \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[U(X^{\pi^*}_T - \xi)] \ \textit{for some } \pi^* \in \mathcal{A}, \mathbb{P} - a.s. \textit{ for all } \mathbb{P} \in \mathcal{P}_H$$

As the minimum condition on K, the property (iii) can be understood as that R^π is a supermartingale under the nonlinear expectation generated by $\mathcal{P}_H$ for every π and R^{π^*} is a martingale under the nonlinear expectation. Then it's not difficult to see that

$$\inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[U(X_T^\pi - \xi)] \leqslant R_0 = \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[U(X_T^{\pi^*} - \xi)] = V^\xi(x). \tag{1.2.1}$$

We consider a financial market which consists of one bond with zero interest rate and d stocks. The price process is given by

$$dS_t = \operatorname{diag}[S_t](b_t dt + dB_t), \ \mathcal{P}_H - q.s.$$

where b is an $\mathbb{R}^d$-valued uniformly bounded stochastic process which is uniformly continuous in ω for the $||\cdot||_\infty$ norm.

It is worth to notice that the volatility is implicitly embedded in the model. Indeed, under each $\mathbb{P}\in\mathcal{P}_H$, we have $dB_s \equiv \widehat{a}_t^{1/2} dW_t^{\mathbb{P}}$ where $W^{\mathbb{P}}$ is a Brownian motion under $\mathbb{P}$. Therefore, $\widehat{a}^{1/2}$ plays the role of volatility under each $\mathbb{P}$ and thus allows us to model the volatility uncertainty.

In the sequel, we show the main result for the exponential utility function which is defined as

$$U(x) = -\exp(-\beta x), \ x\in\mathbb{R} \text{ for } \beta > 0.$$

We have similar results for the power and the logarithmic utility functions.

We define the set of admissible trading strategies as follows

Definition 1.2.1 (Admissible strategies with constraints)**.** *Let A be a closed set in $\mathbb{R}^d$. The set of admissible trading strategies $\mathcal{A}$ consists of all d-dimensional progressively measurable processes, $\pi = (\pi_t)_{0\leqslant t\leqslant T}$ satisfying*

$$\pi \in \mathbb{B}MO \text{ and } \pi_t \in A, \ dt\otimes\mathcal{P}_H - a.e.$$

Usually, when dealing with these type of problems (see for instance [38] and [54]), an exponential uniform integrability assumption is made on the trading strategies. However, we consider instead stronger integrability assumptions of BMO type on the trading strategies. The mathematical reasons behind this are detailed in Chapter 3, however, this also has a financial interpretation. As explained in [43] which adopts the same type of BMO framework, this assumption corresponds to a situation where the market price of risk is assumed to be BMO. Just as in the case of a bounded market price of risk, this implies that the minimum martingale measure is a true probability measure, and therefore there is no arbitrage, in the sense of No Free Lunch with Vanishing Risk.

The investor wants to solve the optimization problem

$$V^\xi(x) := \sup_{\pi\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{Q}}\left[-\exp\left(-\beta(x + \int_0^T \pi_t \frac{dS_t}{S_t} - \xi)\right)\right] \tag{1.2.2}$$

Our main result for robust exponential utility is as follows

Theorem 1.2.1. *Assume that the border of the set A is a C^2 Jordan arc. Then the value function of the optimization problem (1.2.2) is given by*

$$V^{\xi}(x) = -\exp\left(-\beta\left(x - Y_0\right)\right),$$

where Y_0 is defined as the initial value of the unique solution $(Y,Z) \in \mathbb{D}^{\infty}_H \times \mathbb{H}^2_H$ of the following 2BSDE

$$Y_t = \xi - \int_t^T Z_s dB_s - \int_t^T \widehat{F}_s(Z_s)ds + K^{\mathbb{P}}_T - K^{\mathbb{P}}_t,\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H. \tag{1.2.3}$$

The generator has quadratic growth and is defined as follows

$$\widehat{F}_t(\omega, z) := F_t(\omega, z, \widehat{a}_t), \tag{1.2.4}$$

where

$$F_t(\omega, z, a) = -\frac{\beta}{2}\mathrm{dist}^2\left(a^{1/2}z + \frac{1}{\beta}\theta_t(\omega), A_a\right) + z^{'}a^{1/2}\theta_t(\omega) + \frac{1}{2\beta}\left|\theta_t(\omega)\right|^2,\ \textit{for } a \in \mathbb{S}^{>0}_d,$$

with $\theta_t(\omega) = a^{-1/2}b_t(\omega)$ and $A_a := a^{1/2}A = \left\{a^{1/2}b\ :\ b \in A\right\}$.

Moreover, there exists an optimal trading strategy $\pi^* \in \mathcal{A}$ in the sense that for all $\mathbb{P} \in \mathcal{P}_H$

$$\widehat{a}_t^{1/2}\pi^*_t \in \Pi_{A_{\widehat{a}_t}}\left(\widehat{a}_t^{1/2}Z_t + \frac{1}{\beta}\widehat{\theta}_t\right),\quad t \in [0,T],\ \mathbb{P}-\text{a.s.} \tag{1.2.5}$$

where $\widehat{\theta}_t := \widehat{a}_t^{-1/2}b_t$ and $A_{\widehat{a}_t} := \widehat{a}_t^{1/2}A = \left\{\widehat{a}_t^{1/2}b\ :\ b \in A\right\}$.

We also show that the above result can be applied to study the problem of indifference pricing of a contingent claim in the framework of uncertain volatility.

1.3 Second Order Reflected BSDEs

In this part of the thesis, we generalize 2BSDEs theory to the case where there is a lower reflecting obstacle. Reflected backward stochastic differential equations (RBSDEs for short) were introduced by El Karoui et al. [34], followed among others by El Karoui, Pardoux and Quenez in [37] and Bally, Caballero, Fernandez and El Karoui in [3] to study related obstacle problems for PDE's and American options pricing. In this case, the solution Y of the BSDE is constrained to stay above a given obstacle process S. In order to achieve this, a nondecreasing process K is added to the solution

$$\begin{cases} Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t,\ t \in [0,T],\ \mathbb{P}-a.s. \\ Y_t \geqslant S_t,\ t \in [0,T],\ \mathbb{P}-a.s. \\ \int_0^T (Y_s - S_s)dK_s = 0,\ \mathbb{P}-a.s., \end{cases}$$

where the last condition, also known as the Skorohod minimum condition means that the process K only acts when Y reaches the obstacle S. This condition is crucial to obtain the uniqueness of the solution to classical RBSDEs.

Following these pioneering works, many authors have tried to relax the assumptions on the generator of the RBSDE and the corresponding obstacle. Hence, Matoussi [77] and Lepeltier, Matoussi and Xu [70] have extended the existence and uniqueness results to generators with arbitrary growth in y. Then, Kobylanski, Lepeltier, Quenez and Torres [65], Lepeltier and Xu [69] and Bayraktar and Yao [7] studied the case of a generator which is quadratic in z. Similarly, Hamadène [48] and Lepeltier and Xu [68] proved existence and uniqueness when the obstacle is no longer continuous. Cvitanić and Karatzas [25] introduced a new notion of double barrier reflected BSDEs in the case of Lipschitz generators and showed their link with Dynkin games. Later, Hamadène, Lepeltier and Matoussi [50] extended the existence and uniqueness result to the case of continuous generators.

Our aim is to provide a complete theory of existence and uniqueness of solution to 2RBSDEs under the Lipschitz-type hypotheses of [101] on the generator. We show that in this context, the definition of a 2RBSDE with a lower obstacle S is very similar to that of a 2BSDE. We do not need to add another nondecreasing process, unlike in the classical case. The only change required is in the minimum condition that the nondecreasing process K of the 2RBSDE must satisfy. We then establish the link between 2RBSDEs and American contingent claims pricing with UVM.

We start with giving the precise definition of 2RBSDEs and showing how they are connected to classical RBSDEs. As for 2BSDEs with quadratic growth generators, we define F as the corresponding conjugate of a certain map H w.r.t.γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}_d^{>0},$$
$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0, 0).$$

Our main assumptions on the function F are as follows

Assumption 1.3.1. (i) *The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *F is $\mathbb{F}$-progressively measurable in D_{F_t}.*

(iii) *We have the following uniform Lipschitz-type property in y and z*

$$\left|\widehat{F}_t(y, z) - \widehat{F}_t(y^{'}, z^{'})\right| \leqslant C\left(\left|y - y^{'}\right| + \left|\widehat{a}^{1/2}\left(z - z^{'}\right)\right|\right),\ \mathcal{P}_H^\kappa - q.s.$$

for all $(t, y, y^{'}, z, z^{'})$.

(iv) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

Given a process S which will play the role of our lower obstacle. We will always assume S verifies the following properties

(i) S is $\mathbb{F}$-progressively measurable and càdlàg.

(ii) S is uniformly continuous in ω in the sense that for all t

$$|S_t(\omega) - S_t(\widetilde{\omega})| \leqslant \rho\left(\|\omega - \widetilde{\omega}\|_t\right), \ \forall\ (\omega, \widetilde{\omega}) \in \Omega^2$$

for some modulus of continuity ρ and where we define $\|\omega\|_t := \sup_{0 \leqslant s \leqslant t} |\omega(s)|$.

The assumption (i) is quite standard in the classical BSDEs literature; the assumption (ii) is needed to deal with the technicalities induced by the quasi-sure framework.

We denote by $\mathcal{P}_H^\kappa$ the non-dominated class of mutually singular probability measures, where under each $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\widehat{a}$ has positive finite bounds which may depend on $\mathbb{P}$. Then, we shall consider the following 2RBSDE with the lower obstacle S

$$Y_t = \xi - \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \leqslant t \leqslant T,\ \mathcal{P}_H^\kappa - q.s. \tag{1.3.1}$$

Definition 1.3.1. *For $\xi \in \mathcal{L}_H^{2,\kappa}$, we say $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ (the solution space, see Chapter 4 for precise definition) is a solution to the 2RBSDE (1.3.1) if*

- $Y_T = \xi$, $\mathcal{P}_H^\kappa - q.s.$
- $Y_t \geqslant S_t$, $\mathcal{P}_H^\kappa - q.s.$.
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, *the process $K^{\mathbb{P}}$ defined below has nondecreasing paths $\mathbb{P} - a.s.$*

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \leqslant t \leqslant T,\ \mathbb{P} - a.s. \tag{1.3.2}$$

- *We have the following minimum condition*

$$K_t^{\mathbb{P}} - k_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'} - k_T^{\mathbb{P}'}\right],\ 0 \leqslant t \leqslant T,\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{1.3.3}$$

where $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau, \xi), z^{\mathbb{P}}(\tau, \xi), k^{\mathbb{P}}(\tau, \xi))$ denote the unique solution to the following classical RBSDE with obstacle S for any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$,

$$\begin{cases} y_t^{\mathbb{P}} = \xi - \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s + k_\tau^{\mathbb{P}} - k_t^{\mathbb{P}},\ 0 \leqslant t \leqslant \tau,\ \mathbb{P} - a.s. \\ y_t^{\mathbb{P}} \geqslant S_t,\ \mathbb{P} - a.s. \\ \int_0^t \left(y_{s^-}^{\mathbb{P}} - S_{s^-}\right) dk_s^{\mathbb{P}} = 0,\ \mathbb{P} - a.s.,\ \forall t \in [0, T]. \end{cases}$$

The process K plays a double role. Intuitively, K forces Y to stay above the barrier S and it also pushes Y above every $y^{\mathbb{P}}$. To justify this formulation, we can consider the case where the set $\mathcal{P}_H^\kappa$ is reduced to a singleton $\{\mathbb{P}\}$. From the above minimum condition, we know that $K^{\mathbb{P}} - k^{\mathbb{P}}$ is a martingale with finite variation. Since $\mathbb{P}$ satisfies the martingale representation property, this martingale is also continuous, and is therefore a constant. Thus we have

$$0 = k^{\mathbb{P}} - K^{\mathbb{P}},\ \mathbb{P} - a.s.,$$

and the 2RBSDE is equivalent to a standard RBSDE. In particular, we see that the part of $K^{\mathbb{P}}$ which increases only when $Y_{t^-} > S_{t^-}$ is null, which means that $K^{\mathbb{P}}$ satisfies the usual Skorohod condition with respect to the obstacle.

With some additional integrability conditions on $\widehat{F}^0$ and S, we can have a representation formula for a solution to a 2RBSDE via solutions to standard RBSDEs, which in turn implies uniqueness of the solution. This is similar to ones obtained in Theorem 4.4 of [101] and Theorem 2.1 in [90].

Theorem 1.3.1. *Let Assumption 1.3.1 and additional integrability assumptions on $\widehat{F}^0$ and S hold. Assume $\xi \in \mathbb{L}^{2,\kappa}_H$ and that (Y, Z) is a solution to 2RBSDE (1.3.1). Then, for any $\mathbb{P} \in \mathcal{P}^\kappa_H$ and $0 \leqslant t_1 < t_2 \leqslant T$,*

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}^\kappa_H(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \; y^{\mathbb{P}'}_{t_1}(t_2, Y_{t_2}), \ \mathbb{P}-a.s. \tag{1.3.4}$$

Consequently, the 2RBSDE (1.3.1) has at most one solution in $\mathbb{D}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H$.

Now that we have proved the representation (1.3.4), we can show, as in the classical framework, that the solution Y of the 2RBSDE is linked to an optimal stopping problem

Proposition 1.3.1. *Let (Y, Z) be the solution to the above 2RBSDE (1.3.1). Then for each $t \in [0, T]$ and for all $\mathbb{P} \in \mathcal{P}^\kappa_H$*

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}^\kappa_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \; \underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \; \mathbb{E}^{\mathbb{P}'}_t \left[-\int_t^\tau \widehat{F}_s(y^{\mathbb{P}'}_s, z^{\mathbb{P}'}_s)ds + S_\tau \boldsymbol{1}_{\{\tau<T\}} + \xi \boldsymbol{1}_{\{\tau=T\}} \right], \ \mathbb{P}-a.s. \tag{1.3.5}$$

$$= \underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \; \mathbb{E}^{\mathbb{P}}_t \left[-\int_t^\tau \widehat{F}_s(Y_s, Z_s)ds + A^{\mathbb{P}}_\tau - A^{\mathbb{P}}_t + S_\tau \boldsymbol{1}_{\{\tau<T\}} + \xi \boldsymbol{1}_{\{\tau=T\}} \right], \ \mathbb{P}-a.s. \tag{1.3.6}$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times valued in $[t, T]$ and $A^{\mathbb{P}}_t := \int_0^t \boldsymbol{1}_{\{Y_{s^-} > S_{s^-}\}} dK^{\mathbb{P}}_s$ is the part of $K^{\mathbb{P}}$ which only increases when $Y_{s^-} > S_{s^-}$.

It is worth noting here that unlike with classical RBSDEs, considering an upper obstacle in our context is fundamentally different from considering a lower obstacle. Indeed, having a lower obstacle corresponds, at least formally, to add an nondecreasing process in the definition of a 2BSDE. Since there is already an nondecreasing process in that definition, we still end up with an nondecreasing process. However, in the case of an upper obstacle, we would have to add a non-increasing process in the definition, therefore ending up with a finite variation process. This situation thus becomes much more complicated. Furthermore, in this case we conjecture that the above representation of Proposition would hold with a sup-inf instead of a sup-sup, indicating that this situation should be closer to stochastic games than to stochastic control. This is an interesting generalization that we leave for future research.

Then, as for the classical RBSDEs (see Proposition 4.2 in [37]), if we have more regularity on the obstacle S, we can give a more explicit representation for the processes $K^{\mathbb{P}}$. When

S is a semimartingale of the form

$$S_t = S_0 + \int_0^t U_s ds + \int_0^t V_s dB_s + C_t,\ \mathcal{P}_H^\kappa - q.s.$$

For each $\mathbb{P} \in \mathcal{P}_H^\kappa$, there exists a progressively measurable process $(\alpha_t^{\mathbb{P}})_{0 \leqslant t \leqslant T}$ such that $0 \leqslant \alpha \leqslant 1$ and

$$\mathbf{1}_{\{Y_{t^-}=S_{t^-}\}} dK_t^{\mathbb{P}} = \alpha_t^{\mathbb{P}} \mathbf{1}_{\{Y_{t^-}=S_{t^-}\}} \left(\left[\widehat{F}_t(S_t, V_t) - U_t \right]^+ dt + dC_t^- \right), \mathbb{P} - a.s..$$

For existence of a solution, we will generalize the pathwise construction approach of [101] to the reflected case. Let us mention that this proof requires us to extend the existing results on the theory of g-martingales of Peng (see [88]) to the reflected case. Since to the best of our knowledge, those results do not exist in the literature, we prove them in the Appendix in Chapter 4. We are now in position to state the main result of this part

Theorem 1.3.2. *Let $\xi \in \mathcal{L}_H^{2,\kappa}$. Under Assumption 1.3.1 and additional integrability assumptions on $\widehat{F}^0$ and S, there exists a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the* 2RBSDE (1.3.1).

Finally, we use 2RBSDEs introduced previously to study the pricing problem of American contingent claims in a market with volatility uncertainty. The pricing of European contingent claims has already been treated in this context by Avellaneda, Lévy and Paras in [2], Denis and Martini in[27] with capacity theory and more recently by Vorbrink in [110] using the G-expectation framework.

In a financial market with one bond L^0 with interest rate r_t and one risky asset L, whose dynamic is given by

$$\frac{dL_t}{L_t} = \mu_t dt + dB_t,\ \mathcal{P}_H^\kappa - q.s.,$$

we consider an American contingent claim whose payoff at a stopping time $\nu \geqslant t$ is

$$\tilde{S}_\nu = S_\nu \mathbf{1}_{[\nu<T]} + \xi \mathbf{1}_{[\nu=T]}.$$

Then with some assumptions on r, μ and S which ensure the existence of a solution to a 2RBSDE, we have that, for $\xi \in \mathcal{L}_H^{2,\kappa}$, a superhedging price for the contingent claim is

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} Y_t^{\mathbb{P}'},\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa,$$

where $Y_t^{\mathbb{P}'}$ is the price at time t of the same contingent claim in the complete market, with underlying probability measure $\mathbb{P}'$. The process Y_t is the solution to a 2RBSDE with a Lipschitz generator which depends on r and μ.

Furthermore, we have, for all ε, the stopping time $D_t^\varepsilon = \inf\{s \geqslant t, Y_s \leqslant S_s + \varepsilon\} \wedge T$ is ε-optimal after t. Besides, for all $\mathbb{P}$, if we consider the stopping times $D_t^{\varepsilon,\mathbb{P}} = \inf\left\{s \geqslant t, Y_s^{\mathbb{P}} \leqslant S_s + \varepsilon\right\} \wedge T$, which are ε-optimal for the American contingent claim under each $\mathbb{P}$, then as a consequence of the representation formula, we have

$$D_t^\varepsilon \geqslant D_t^{\varepsilon,\mathbb{P}},\ \mathbb{P} - a.s. \tag{1.3.7}$$

1.4 Second Order BSDEs with Jumps

From the literature, we know that in the case of a filtered probability space generated by both a Brownian motion W and a Poisson random measure μ with compensator ν, one can consider the following natural generalization of BSDE (1.0.1) to the case with jumps. We say that (Y, Z, U) is a solution of the BSDE with jumps (BSDEJ for short) with generator f and terminal condition ξ if for all $t \in [0, T]$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}^d \setminus \{0\}} U_s(x)(\mu - \nu)(ds, dx), \ \mathbb{P} - a.s. \tag{1.4.1}$$

Tang and Li [106] were the first to prove existence and uniqueness of a solution for (1.4.1) with a fixed point argument in the case where f is Lipschitz in (y, z, u). Barles et al. [5] studied the link of those BSDEJs with viscosity solutions of integral-partial differential equations. Hamadène and Ouknine [51] have considered one reflecting barrier BSDEJs. They showed existence and uniqueness of the solution when the reflecting barrier has only inaccessible jumps, i.e., jumps which come only from the Poisson part. Hamadène and Ouknine [52] and Essaky [39] then respectively dealt with reflected BSDEJs when the reflecting processes are càdlàg. In general, in contrary to BSDEs, there is no comparison theorem for BSDEJs with only Lipschitz generators. One needs stronger assumptions. Royer in [95] proved a comparison theorem and studied nonlinear expectations related to BSDEs with jumps which extends Peng's g-expectation framework to the jump case. Crépey and Matoussi [24] also provided *a priori* estimates and comparison theorem for reflected and doubly reflected BSDEJs. [83] studied a special BSDEJ with quadratic growth related to the problem of exponential utility maximization under constraint. Recently, [36] adopted a forward approach as in [6] to prove existence of quadratic BSDEJs with unbounded terminal condition.

In this part of the thesis, we generalize 2BSDEs to the jump case. We can interpret these equations as standard BSDEJs, under both volatility and jump measure uncertainty.

On the Skorohod space, we define the continuous part of the canonical process B, noted by B^c, and its purely discontinuous part, noted by B^d, both local martingales under a local martingale measure. Such local martingale measures are obtained by using the notion of martingale problem for semimartingales with general characteristics, as defined in the book by Jacod and Shiryaev [56]. We then associate to the jumps of B a counting measure μ_{B^d}.

To define correctly the notion of second order backward SDEs with jumps (2BSDEJs), an important issue is the possibility to aggregate both the quadratic variation $[B, B]$ of the canonical process and the compensated jump measure associated to B^d, in the following sense of [103] and [23]:

Let $\mathcal{P}$ be a set of non necessarily dominated probability measures and let $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ be a family of random variables indexed by $\mathcal{P}$. An *aggregator* of the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ is a random variable $\hat{X}$ such that

$$\hat{X} = X^{\mathbb{P}}, \ \mathbb{P} - \text{a.s, for every } \mathbb{P} \in \mathcal{P}.$$

We know that the quadratic variation $[B, B]$ can be aggregated as a consequence of the results from Bichteler [9], Karandikar [58], or more recently Nutz [86]. However, the predictable compensator is usually obtained by the Doob-Meyer decomposition of the submartingale $[B, B]$. It is therefore clear that this compensator depends explicitly on the underlying probability measure, and it is not clear at all whether an aggregator always exists or not. This is a main difference with the continuous case.

Soner, Touzi and Zhang, motivated by the study of stochastic target problems under volatility uncertainty, obtained in [103] an aggregation result for a family of probability measures corresponding to the laws of some continuous martingales on the canonical space $\Omega = \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$, under a *separability* assumption on the quadratic variations (see their definition 4.8) and an additional *consistency* condition (which is usually only necessary) for the family to aggregate.

In our context, we follow the spirit of [103] and restrict our set of probability measures (by adding an analogous separability condition for jump measures) in order to generalize some of their results in [103] to the case of processes with jumps. We characterize the family of probability measures where we can aggregate both the quadratic variation and the compensated jump measure.

After addressing this aggregation issue, we are in a position to prove the wellposedness of 2BSDEJ under a set of probability measures, denoted by $\mathcal{P}_{\tilde{\mathcal{A}}}$, which has the required characterization. We give a pathwise definition of the process $\widehat{a}$, which is an aggregator for the density of the quadratic variation of the continuous part B^c,

$$\widehat{a}_t := \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left(\langle B^c \rangle_t - \langle B^c \rangle_{t-\varepsilon} \right),$$

and define a process $\widehat{\nu}$, which is an aggregator of the predictable compensators associated to the jump measure μ_{B^d}

$$\widehat{\nu}_t(A) = \nu_t^{\mathbb{P}}(A), \text{ for every } \mathbb{P} \in \tilde{\mathcal{P}_{\mathcal{A}}}. \tag{1.4.2}$$

We then denote

$$\widetilde{\mu}_{B^d}(dt, dx) := \mu_{B^d}(dt, dx) - \widehat{\nu}_t(dx)dt.$$

The generator F, defined as the convex conjugate of a given map, verifies the usual assumptions in t and ω as in the 2BSDEs framework and the uniform Lipschitz assumption in y and z. In the variable u, we need an assumption similar to that in Royer [95].

For all $(t, \omega, y, z, u^1, u^2, a, \nu)$, there exist two processes γ and γ' such that

$$\text{(i)} \int_E \left(u^1(e) - u^2(e)\right) \gamma_t(e) \nu(de) \leqslant F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu),$$

$$\text{(ii)} \; F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu) \leqslant \int_E \left(u^1(e) - u^2(e)\right) \gamma'_t(e) \nu(de)$$

with $c_1(1 \wedge |x|) \leqslant \gamma_t(x) \leqslant c_2(1 \wedge |x|)$ where $c_1 \leqslant 0, \; 0 \leqslant c_2 < 1$,
and $c'_1(1 \wedge |x|) \leqslant \gamma'_t(x) \leqslant c'_2(1 \wedge |x|)$ where $c'_1 \leqslant 0, \; 0 \leqslant c'_2 < 1$.

Then, with assumption (i), we have a comparison theorem which is crucial to have a representation for the Y-part of a solution. We need assumption (ii) to prove the minimum condition satisfied by K for the existence result.

As in [101] we fix a constant $\kappa \in (1,2]$ and restrict the probability measures in $\mathcal{P}_H^\kappa \subset \mathcal{P}_{\tilde{\mathcal{A}}}$. We shall consider the following 2BSDEJ, for $0 \leqslant t \leqslant T$ and $\mathcal{P}_H^\kappa$-q.s.

$$Y_t = \xi - \int_t^T F_s(Y_s, Z_s, U_s, \widehat{a}_s, \widehat{\nu}_s)ds - \int_t^T Z_s dB_s^c - \int_t^T \int_E U_s(x)\tilde{\mu}_{B^d}(ds,dx) + K_T - K_t. \tag{1.4.3}$$

Similar to 2BSDEs, we say (Y, Z, U) is a solution to the 2BSDEJ (1.4.3) if the equation holds true under each $\mathbb{P} \in \mathcal{P}_H^\kappa$ with a nondecreasing process $K^{\mathbb{P}}$ and the family $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ satisfies the minimum condition.

With a generalization of the comparison theorem and the minimum condition on K, as usual, we have a representation formula for the Y-part of a solution.

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \; y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \;\; \mathbb{P} - a.s., \tag{1.4.4}$$

where $y^{\mathbb{P}'}$ is the solution to the standard BSDE with the same generator under $\mathbb{P}' \in \mathcal{P}_H^\kappa$.

For the existence, we generalize the usual approach in 2BSDEs theory to the jump case. We construct a solution pathwise when terminal condition is in a regular space, then by passing to limit, we show existence of a solution for terminal condition in its closure under a certain norm.

As an application of the above results, we study a problem of robust utility maximization under model uncertainty, which affects both the volatility process and the jump measure. We consider a financial market consisting of one riskless asset, whose price is assumed to be equal to one for simplicity, and one risky asset whose price process $(S_t)_{0 \leqslant t \leqslant T}$ is assumed to follow a jump-diffusion with regular coefficients

$$\frac{dS_t}{S_{t^-}} = b_t dt + dB_t^c + \int_E \beta_t(x)\mu_{B^d}(dt,dx). \tag{1.4.5}$$

The problem of the investor in this financial market is to maximize his expected exponential utility under model uncertainty from his total wealth $X_T^\pi - \xi$, where ξ is a liability at time T which is a $\mathcal{F}_T$-measurable random variable. The trading strategies are supposed to take value in some compact set C. Then the value function V of the maximization problem can be written as

$$\begin{aligned} V^\xi(x) :&= \sup_{\pi \in \mathcal{C}} \inf_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-\eta\left(X_T^\pi - \xi\right)\right)\right] \\ &= -\inf_{\pi \in \mathcal{C}} \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\eta\left(X_T^\pi - \xi\right)\right)\right]. \end{aligned} \tag{1.4.6}$$

We follow the ideas of the *martingale optimality principle* approach adapted to the nonlinear framework as in Chapter 3. We prove that the value function of the optimization problem 1.4.6 is given by

$$V^{\xi}(x) = -e^{-\eta x} Y_0,$$

where Y_0 is defined as the initial value of the unique solution (Y, Z, U) of the 2BSDEJ with terminal condition $e^{\eta\xi}$ and the generator

$$F_t(y, z, u, a, \nu) := -\inf_{\pi \in C} \left\{ (-\eta b_t + \frac{\eta^2}{2}\pi a)\pi y - \eta \pi a z + \int_E \left(e^{-\eta\pi\beta_t(x)} - 1\right)(y + u(x))\nu(dx) \right\}.$$

Moreover, there exists an optimal trading strategy π^* realizing the infimum above.

Finally, as in Lim and Quenez [73] for BSDEs, by making a change of variables, we derive existence and uniqueness of a solution to a 2BSDEJ with quadratic growth from this 2BSDEJ with a Lipschitz generator.

Recall that Pardoux and Peng [87] proved that if the randomness in g and ξ is induced by the current value of a state process defined by a forward stochastic differential equation, then the solution to a BSDE could be linked to the solution of a semilinear PDE by means of a generalized Feynman-Kac formula. Soner, Touzi and Zhang [101] also introduced the second order backward SDEs in a non dominated framework. Their equations generalize the point of view of Pardoux and Peng, in the sense that they are connected to the larger class of fully nonlinear PDEs. In this context, the 2BSDEJs are the natural candidates for a probabilistic solution of fully nonlinear integro-differential equations. This is the purpose of our accompanying paper [62].

1.5 Numerical Implementation

In this part of the thesis, I present some practical work realized during an internship during the first year of this PhD study. The subject is Monte Carlo method for options pricing with UVM . The objective is not to prove convergence results of new numerical schemes, but to implement the existing schemes (see Guyon and Henry-Labordère [47]), and to test and possibly make improvement in practice. This work allowed me to understand better these schemes and to be familiar with them. For future research, I would like to suggest a purely probabilistic scheme with the new formulation of 2BSDEs in view (see [101]).

As explained in El Karoui, Peng and Quenez [33] and in El Karoui, Hamadène and Matoussi [35], BSDEs can be used for the pricing of contingent claims by replication in a complete market (with a linear generator f) and more interesting in imperfect market (with a Lipschitz generator f). More precisely, Y corresponds to the value of the replication portfolio and Z is related to the hedging strategy. Since the analytical solution exists to BSDEs only in few case, numerical resolution is important for the application of BSDEs theory in practice in mathematical finance. Moreover, due to the link between BSDEs and semilinear PDEs, numerical resolution of BSDEs is also useful to provide probabilistic numerical methods to solve PDEs. These methods are alternative to finite difference ones, and they are more efficient in high-dimensional case. However, compare to the large amount literature dedicated to the mathematical analysis of BSDEs, only a few numerical methods have been proposed to solve them. We can refer to Bouchard and Touzi [16], Zhang [111], Gobet et al. [45] among others.

We consider the following (decoupled) forward-backward stochastic differential equations on the time interval $[0, 1]$:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad dY_t = f(t, X_t, Y_t, Z_t)dt - Z_t \cdot dW_t$$

$$X_0 = x \text{ and } Y_1 = g(X_{[0,1]})$$

Zhang [111] proved a regularity result on Z, which allows the use of a regular deterministic time mesh. Therefore by discretizing the continuous processes of BSDE and taking the conditional expectation of both sides of equations (resp. first multiplying both sides by Brownian increment ΔW, then taking the conditional expectation), one can compute Y (resp. Z) backwardly. The following is the complete scheme, for $0 = t_0 < t_1 < \cdots < t_n = 1$

$$\begin{cases} Y^\Delta_{t_n} = g^\Delta(X^\Delta_{\{t_0,\cdots,t_n\}}) \\ Y^\Delta_{t_{i-1}} = \mathbb{E}_{i-1}\left[Y^\Delta_{t_i}\right] + f\left(t_{i-1}, X^\Delta_{t_{i-1}}, Y^\Delta_{t_{i-1}}, Z^\Delta_{t_{i-1}}\right)\Delta t_i \\ Z^\Delta_{t_{i-1}} = \dfrac{1}{\Delta t_i}\mathbb{E}_{i-1}\left[Y^\Delta_{t_i}\Delta W_{t_i}\right] \end{cases} \tag{1.5.1}$$

The key point of this scheme is to compute the conditional expectations. In [111], the complexity to compute the conditional expectations becomes very large in multidimensional problems, like in the case of finite difference schemes for PDEs. To better deal with high-dimensional problems, Bouchard and Touzi [16] proposed a Monte Carlo approach when the terminal condition is non-path-dependent (that is $Y_1 = g(X_1)$). They suggested to use a general regression operator found with Malliavin calculus which, however, requires multiple sets of paths. Later, Gobet et al. [45] developed an approach based on Monte Carlo regression on a finite basis of functions, which was first introduced by Longstaff and Schwartz [74] for the pricing of Bermuda options. Their approach is more efficient, because it requires only one set of paths to approximate all regression operators.

Numerical resolution of BSDEs can be applied to numerically solve only semilinear PDEs. More recently, some authors proposed several Monte Carlo numerical schemes for fully nonlinear PDEs. Theses schemes are largely inspired by those for BSDEs.

In their first formulation of 2BSDEs, Cheridito et al. [22] suggests an adaptation of BSDEs numerical scheme to the 2BSDEs case. Inspired by Scheme Cheridito et al., Fahim et al. [41] gives a new scheme without appealing to the theory of 2BSDEs. With uncertain volatility models, the pricing PDE derived in Avellaneda et al. [2] is fully nonlinear. In this particular case, Guyon and Henry-Labordère [47] improves the two precedent schemes without using the theory of 2BSDE. For path-dependent options, these schemes can also be applied with some modifications and by using results obtained in Gobet et al. [45].

For the pricing of Bermuda options, Bouchard and Warin [18] suggests to construct confidence intervals for the true price, one bound from a backward computation and the other one from a backward-forward computation. Both quantities can be computed at the same time with almost no additional cost. Their construction can be adopted in the above probabilistic numerical methods for fully nonlinear PDEs. A small confidence interval should reveal a good approximation of the exact price, while a large confidence interval should be a sign that the estimator was poor.

We implement Scheme Guyon and Henry-Labordère [47] for pricing options, with both backward computations and backward-forward computations. We also suggest some techniques to improve the scheme in practice. From the numerical test results, we generally observe that the Monte Carlo method performs well for non-path-dependent options and can provide prices with good precision for path-dependent ones. Moreover, the pricing precision depends essentially on the quality of the approximation of conditional expectations by regression. In order to get more precise results with this method, we should improve the approximation of conditional expectations by using better regression procedure, suitable control variates and/or non-parametric regressions in higher dimension. In particular, special knowledge of financial products could be used to have better result.

1.6 Work in preparation and future research perspectives

We end the introduction by presenting some work in preparation and future research topics.

First, we are interested in Sobolev solutions of the obstacle problems associated to partial integral-differential equations (PIDEs for short). We give probabilistic interpretation for these solutions via Lipschitz RBSDEs with jumps by developing a stochastic flow method which has been introduced by Bally and Matoussi in [4] in the study of weak solution of stochastic partial differential equations. In another work, we prove existence and uniqueness of a solution to BSDEs with jumps with quadratic growth generators by a fixed point argument as in Tevzadze [107], and we generalize the results of g-nonlinear expectations related to BSDEs with jumps in Royer [95] to the case of quadratic growth. Last but not least, we study the connection between 2BSDEJs and fully nonlinear PIDEs.

For future research, one topic is about 2RBSDEs with one upper obstacle and with double obstacles. This will allow us to study problems of stochastic games with volatility uncertainty. Other possibility is to extend 2BSDEJs to the case of quadratic growth generators and the case with obstacles. For the existence of a solution to 2BSDEs with quadratic growth and 2RBSDEs, it is also interesting to have another proof based on approximation techniques similar to those used in the classical BSDEs literature. For that, we need general monotone convergence theorem and dominated convergence theorem for quasi-sure stochastic analysis. This approach should allow us to prove the wellposedness of these classes of 2BSDEs under weaker assumptions. The last topic is about numerical method. With the new formulation of 2BSDEs and 2BSDEJs in view, it will be interesting to find purely probabilistic schemes for fully nonlinear PDEs and PIDEs.

Chapitre 2

Second Order BSDEs with Quadratic Growth

2.1 Introduction

In this chapter, we provide an existence and uniqueness result for 2BSDEs with quadratic growth generators. The outline is as follows. After introducing the framework of 2BSDEs and the main assumptions on the generator in Section 2.2, we give a stochastic representation for the Y-part of a solution in Section 2.3. This representation then implies the uniqueness of the solution. In Section 2.5, we use the method introduced by Soner, Touzi and Zhang [101] to construct the solution to the quadratic 2BSDE path by path. Finally, in Section 2.7, we extend the results of Soner, Touzi and Zhang on the connections between fully nonlinear PDEs and 2BSDEs to the quadratic case. This chapter is based on [92].

In this chapter, we propose two very different methods to prove the wellposedness in the 2BSDE case. First, we recall some notations in Section 2.2 and prove a uniqueness result in Section 2.3 by means of a priori estimates and a representation of the solution inspired by the stochastic control theory. Then, Section 2.4 is devoted to the study of approximation techniques for the problem of existence of a solution. We advocate that since we are working under a family of non-dominated probability measures, the monotone or dominated convergence theorem may fail. This is a major problem, and we spend some time explaining why, in general, the classical methods using exponential changes fail for 2BSDEs. Nonetheless, using very recent results of Briand and Elie [14], we are able to show a first existence result using an approximation method. Then in Section 2.3, we use a completely different method introduced by Soner, Touzi and Zhang [101] to construct the solution to the quadratic 2BSDE path by path. Next, we use these results in Section 2.6 to study an application of 2BSDEs with quadratic growth to robust risk-sensitive control problems. Finally, in Section 2.7, we extend the results of Soner, Touzi and Zhang [101] on the connections between fully non-linear PDEs and 2BSDEs to the quadratic case. This chapter is based on [92].

2.2 Preliminaries

Let $\Omega := \left\{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\right\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0 \leqslant t \leqslant T}|\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}$ the filtration generated by B, and $\mathbb{F}^+ := \left\{\mathcal{F}_t^+\right\}_{0 \leqslant t \leqslant T}$ the right limit of $\mathbb{F}$. We first recall the notations introduced in [101].

2.2.1 A first set of probability measures

A probability measure $\mathbb{P}$ is said to be a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. By Karandikar [58], tit is known that there exists an $\mathbb{F}$-progressively measurable process, denoted as $\int_0^t B_s dB_s$, which coincides with the Itô's integral, $\mathbb{P}-a.s.$ for all local martingale measure $\mathbb{P}$. In addition, this provides a pathwise definition of

$$\langle B\rangle_t := B_t B_t^T - 2\int_0^t B_s dB_s^T \text{ and } \widehat{a}_t := \limsup_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\left(\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}\right),$$

where ^{T} denotes the transposition and the lim sup is componentwise.

Let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \text{ is absolutely continuous in } t \text{ and } \widehat{a} \text{ takes values in } \mathbb{S}_d^{>0},\ \mathbb{P}-a.s. \tag{2.2.1}$$

where $\mathbb{S}_d^{>0}$ denotes the space of all $d\times d$ real valued positive definite matrices.

As in [101], we concentrate on the subclass $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ consisting of all probability measures

$$\mathbb{P}^\alpha := \mathbb{P}_0\circ (X^\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s,\ t\in[0,T],\ \mathbb{P}_0 - a.s. \tag{2.2.2}$$

for some $\mathbb{F}$-progressively measurable process α taking values in $\mathbb{S}_d^{>0}$ and satisfying $\int_0^T |\alpha_s|\, ds < +\infty$ $\mathbb{P}_0 - a.s.$ We recall from [102] that every $\mathbb{P}\in\overline{\mathcal{P}}_S$ satisfies the Blumenthal zero-one law and the martingale representation property.

Notice that the set $\overline{\mathcal{P}}_S$ is bigger that the set $\widetilde{\mathcal{P}}_S$ introduced in [90], which is defined by

$$\widetilde{\mathcal{P}}_S := \left\{\mathbb{P}^\alpha \in \overline{\mathcal{P}}_S,\ \underline{a}\leqslant\alpha\leqslant\bar{a},\ \mathbb{P}_0 - a.s.\right\}, \tag{2.2.3}$$

for fixed matrices $\underline{a}$ and $\bar{a}$ in $\mathbb{S}_d^{>0}$.

2.2.2 The Generator and the final set $\mathcal{P}_H$

Before defining the spaces under which we will be working or defining the 2BSDE itself, we first need to restrict one more time our set of probability measures, using explicitely the generator of the 2BSDE.

Following the PDE intuition recalled in the Introduction 1, let us first consider a map $H_t(\omega,y,z,\gamma) : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times D_H\to\mathbb{R}$, where $D_H\subset\mathbb{R}^{d\times d}$ is a given subset containing 0. As expected, we define its Fenchel-Legendre conjugate w.r.t.γ by

$$F_t(\omega,y,z,a) := \sup_{\gamma\in D_H}\left\{\frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega,y,z,\gamma)\right\} \text{ for } a\in\mathbb{S}_d^{>0}$$
$$\widehat{F}_t(y,z) := F_t(y,z,\widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0,0).$$

We denote by $D_{F_t(y,z)}$ the domain of F in a for a fixed (t,ω,y,z), and as in [101] we restrict the probability measures in $\mathcal{P}_H\subset\overline{\mathcal{P}}_S$

Definition 2.2.1. *$\mathcal{P}_H$ consists of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a}_{\mathbb{P}} \leqslant \widehat{a} \leqslant \bar{a}_{\mathbb{P}},\ dt \times d\mathbb{P} - a.s. \text{ for some } \underline{a}_{\mathbb{P}}, \bar{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0},\ \text{ and } \widehat{a}_t \in D_{F_t(0,0)},\ dt \times d\mathbb{P} - a.s..$$

Remark 2.2.1. *The restriction to the set $\mathcal{P}_H$ obeys two imperatives. First, since $\widehat{F}$ is destined to be the generator of our 2BSDE, we obviously need to restrict ourselves to probability measures such that $\widehat{a}_t \in D_{F_t(0,0)}$. Moreover, we also restrict the measures considered to the ones such that the density of the quadratic variation of B is bounded to ensure that B is actually a true martingale under each of those probability measures. This will be important to obtain a priori estimates.*

Finally, we recall

Definition 2.2.2. *We say that a property holds $\mathcal{P}_H$-quasi surely ($\mathcal{P}_H - q.s.$ for short) if it holds $\mathbb{P} - a.s.$ for all $\mathbb{P} \in \mathcal{P}_H$.*

2.2.3 Assumptions

We now state our main assumptions on the function F which will be our main interest in the sequel

Assumption 2.2.1. (i) *$\mathcal{P}_H$ is not empty, and the domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *In D_{F_t}, F is $\mathbb{F}$-progressively measurable.*

(iii) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

(iv) *F is continuous in z and has the following growth property. There exists $(\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$ such that*

$$\left|\widehat{F}_t(y,z)\right| \leqslant \alpha + \beta\,|y| + \frac{\gamma}{2}\left|\widehat{a}^{1/2} z\right|^2, \mathcal{P}_H - q.s., \text{ for all } (t,y,z).$$

(v) *F is C^1 in y and C^2 in z, and there are constants r and θ such that for all (t,y,z),*

$$|D_y \widehat{F}_t(y,z)| \leqslant r,\ |D_z \widehat{F}_t(y,z)| \leqslant r + \theta\left|\widehat{a}^{1/2} z\right|,$$
$$|D^2_{zz} \widehat{F}_t(y,z)| \leqslant \theta,\ \mathcal{P}_H - q.s..$$

Remark 2.2.2. *Let us comment on the above assumptions. Assumptions 2.2.1* (i) *and* (iii) *are taken from [101] and are needed to deal with the technicalities induced by the quasi-sure framework. Assumptions 2.2.1* (ii) *and* (iv) *are quite standard in the classical BSDE literature. Finally, Assumption 2.2.1* (v) *was introduced by Tevzadze in [107] for quadratic BSDEs. It allowed him to prove existence of quadratic BSDEs through fixed point arguments. This is this consequence which will be used for technical reasons in Section 2.5.*

However, it was also showed in [107], that if both the terminal condition and $\widehat{F}^0$ are small enough, then Assumption 2.2.1 (v) *can be replaced by a weaker one. We will therefore sometimes consider*

Assumption 2.2.2. *Let* (i), (ii),(iii) and (iv) of Assumption 2.2.1 hold and

(v) We have the following "local Lipschitz" assumption in z, $\exists\mu > 0$ and a progressively measurable process $\phi \in \mathbb{B}\mathrm{MO}(\mathcal{P}_\mathrm{H})$ such that for all $(t, y, z, z^{'})$,

$$\left|\widehat{F}_t(y,z) - \widehat{F}_t(y,z^{'}) - \phi_t.\widehat{a}^{1/2}(z-z^{'})\right| \leqslant \mu\widehat{a}^{1/2}\left|z-z^{'}\right|\left(\left|\widehat{a}^{1/2}z\right| + \left|\widehat{a}^{1/2}z^{'}\right|\right)$$
$$\mathcal{P}_H - q.s.$$

(vi) We have the following uniform Lipschitz-type property in y

$$\left|\widehat{F}_t(y,z) - \widehat{F}_t(y^{'},z)\right| \leqslant C\left|y-y^{'}\right|, \mathcal{P}_H - q.s., \text{ for all } (y,y^{'},z,t).$$

Furthermore, we observe that our subsequent proof for uniqueness of a solution of our quadratic 2BSDE only use Assumption 2.2.2.

Remark 2.2.3. *Assumption 2.2.1*(i) *implies that $\widehat{a}$ always belongs to $D_{F_t(y,z)}$. Moreover, by Assumption 2.2.1*(iv)*, we have that $\widehat{F}_t^0$ is actually bounded, so the strong integrability condition*

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}_t^0\right|^\kappa dt\right)^{\frac{2}{\kappa}}\right] < +\infty,$$

with a constant $\kappa \in (1,2]$ introduced in [101] is not needed here.

2.2.4 Spaces of interest

We now recall from [101] the spaces and norms which will be needed for the formulation of 2BSDEs and add some specific spaces which are linked to our quadratic growth framework.

For $p \geqslant 1$, L_H^p denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|_{L_H^p}^p := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty.$$

In the case $p = +\infty$ we define similarly the space of random variables which are bounded quasi-surely and take as a norm

$$\|\xi\|_{L_H^\infty} := \sup_{\mathbb{P}\in\mathcal{P}_H} \|\xi\|_{L^\infty(\mathbb{P})}.$$

$\mathbb{H}_H^p$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{H}_H^p}^p := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\widehat{a}_t^{1/2}Z_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

$\mathbb{D}_H^p$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H - q.s. \text{ càdlàg paths, and } \|Y\|_{\mathbb{D}_H^p}^p := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T} |Y_t|^p\right] < +\infty.$$

In the case $p = +\infty$ we define

$$\|Y\|_{\mathbb{D}^\infty_H} := \sup_{0 \leqslant t \leqslant T} \|Y_t\|_{L^\infty_H}.$$

For each $\xi \in L^1_H$, $\mathbb{P} \in \mathcal{P}_H$ and $t \in [0,T]$ denote

$$\mathbb{E}^{H,\mathbb{P}}_t[\xi] := \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{}^{\mathbb{P}}\mathbb{E}^{\mathbb{P}'}_t[\xi] \text{ where } \mathcal{P}_H(t^+,\mathbb{P}) := \left\{\mathbb{P}' \in \mathcal{P}_H : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}^+_t\right\}.$$

Here $\mathbb{E}^{\mathbb{P}}_t[\xi] := E^{\mathbb{P}}[\xi|\mathcal{F}_t]$. Then we define for each $p \geqslant 1$,

$$\mathbb{L}^p_H := \left\{\xi \in L^p_H : \|\xi\|_{\mathbb{L}^p_H} < +\infty\right\} \text{ where } \|\xi\|^p_{\mathbb{L}^p_H} := \sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0 \leqslant t \leqslant T}\left(\mathbb{E}^{H,\mathbb{P}}_t[|\xi|]\right)^p\right].$$

In the case $p = +\infty$ the natural generalization of the norm $\mathbb{L}^p_H$ is the norm L^∞_H introduced above. Therefore, we will use the latter in order to be consistent with the notations of [101].

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}^p_H := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}^p_H}, \text{ for every } p \geqslant 1.$$

2.2.4.1 The space $\mathbb{B}\mathrm{MO}(\mathcal{P}_\mathrm{H})$ and important properties

It is a well known fact that the Z component of the solution of a quadratic BSDE with a bounded terminal condition belongs to the so-called BMO space. Since this link will be extended and used intensively throughout the paper, we will recall some results and definitions for the BMO space, and then extend them to our quasi-sure framework. We first recall (with a slight abuse of notation) the definition of the BMO space for a given probability measure $\mathbb{P}$.

Definition 2.2.3. BMO($\mathbb{P}$) *denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with*

$$\|Z\|_{\mathrm{BMO}(\mathbb{P})} := \sup_{\tau \in \mathcal{T}^T_0}\left\|\mathbb{E}^{\mathbb{P}}_\tau\left[\int_\tau^T |\widehat{a}^{1/2}_t Z_t|^2 dt\right]\right\|_\infty < +\infty,$$

where $\mathcal{T}^T_0$ is the set of $\mathcal{F}_t$ stopping times taking their values in $[0,T]$.

We also recall the so called energy inequalities (see [59] and the references therein). Let $Z \in \mathrm{BMO}(\mathbb{P})$ and $p \geqslant 1$. Then we have

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{a}^{1/2}_s Z_s\right|^2 ds\right)^p\right] \leqslant 2p!\left(4\|Z\|^2_{\mathbb{H}^2_{BMO}}\right)^p. \tag{2.2.4}$$

The extension to a quasi-sure framework is then naturally given by the following space.

$\mathbb{B}\mathrm{MO}(\mathcal{P}_\mathrm{H})$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{B}\mathrm{MO}(\mathcal{P}_\mathrm{H})} := \sup_{\mathbb{P} \in \mathcal{P}_H} \|Z\|_{\mathrm{BMO}(\mathbb{P})} < +\infty.$$

We say that $\int_0^\cdot Z_s dB_s$ is a $\mathbb{BMO}(\mathcal{P}_{\rm H})$ martingale if $Z \in \mathbb{BMO}(\mathcal{P}_{\rm H})$.

The main interest of the BMO spaces is that if a process Z belongs to it, then the stochastic integral $\int_0^\cdot Z_s dB_s$ is a uniformly integrable martingale, which in turn allows us to use it for changing the probability measure considered via Girsanov's Theorem. The two following results give more detailed results in terms of L^r integrability of the corresponding Doléans-Dade exponentials.

Lemma 2.2.1. *Let $Z \in \mathbb{BMO}(\mathcal{P}_{\rm H})$. Then there exists $r > 1$, such that*

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\left(\mathcal{E}\left(\int_0^\cdot Z_s dB_s\right)\right)^r\right] < +\infty.$$

Proof. By Theorem 3.1 in [59], we know that if $\|Z\|_{\mathbb{BMO}(\mathbb{P})} \leqslant \Phi(r)$ for some one-to-one function Φ from $(1, +\infty)$ to $\mathbb{R}_+^*$, then $\mathcal{E}\left(\int_0^\cdot Z_s dB_s\right)$ is in $L^r(\mathbb{P})$. Here, since $Z \in \mathbb{BMO}(\mathcal{P}_{\rm H})$, the same r can be used for all the probability measures. □

Lemma 2.2.2. *Let $Z \in \mathbb{BMO}(\mathcal{P}_{\rm H})$. Then there exists $r > 1$, such that for all $t \in [0,T]$*

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}_t^{\mathbb{P}}\left[\left(\frac{\mathcal{E}\left(\int_0^t Z_s dB_s\right)}{\mathcal{E}\left(\int_0^T Z_s dB_s\right)}\right)^{\frac{1}{r-1}}\right] < +\infty.$$

Proof. This is a direct application of Theorem 2.4 in [59] for all $\mathbb{P} \in \mathcal{P}_H$. □

We emphasize that the two previous Lemmas are absolutely crucial to our proof of uniqueness and existence. Besides, they will also play a major role in Chapter 3.

2.2.5 The definition of the 2BSDE

Everything is now ready to define the solution of a 2BSDE. We shall consider the following 2BSDE, which was first defined in [101]

$$Y_t = \xi - \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \leqslant t \leqslant T,\ \mathcal{P}_H - q.s. \tag{2.2.5}$$

Definition 2.2.4. *We say $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ is a solution to 2BSDE (2.2.5) if :*

- $Y_T = \xi$, $\mathcal{P}_H - q.s.$
- *For all $\mathbb{P} \in \mathcal{P}_H$, the process $K^{\mathbb{P}}$ defined below has nondecreasing paths $\mathbb{P} - a.s.$*

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \leqslant t \leqslant T,\ \mathbb{P} - a.s. \tag{2.2.6}$$

- *The family $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\right\}$ satisfies the minimum condition*

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\right],\ 0 \leqslant t \leqslant T,\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H. \tag{2.2.7}$$

Moreover if the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ *can be aggregated into a universal process* K*, we call* (Y, Z, K) *a solution of 2BSDE* (2.2.5)*.*

Remark 2.2.4. *Let us comment on this definition. As already explained, the PDE intuition leads us to think that the solution of a 2BSDE should be a supremum of solution of standard BSDEs. Therefore for each* $\mathbb{P}$*, the role of the non-decreasing process* $K^{\mathbb{P}}$ *is in some sense to "push" the process* Y *to remain above the solution of the BSDE with terminal condition* ξ *and generator* $\widehat{F}$ *under* $\mathbb{P}$*. In this regard, 2BSDEs share some similarities with reflected BSDEs.*

Pursuing this analogy, the minimum condition (2.2.7) *tells us that the processes* $K^{\mathbb{P}}$ *act in a "minimal" way (exactly as implied by the Skorohod condition for reflected BSDEs), and we will see in the next Section that it implies uniqueness of the solution. Besides, if the set* $\mathcal{P}_H$ *was reduced to a singleton* $\{\mathbb{P}\}$*, then* (2.2.7) *would imply that* $K^{\mathbb{P}}$ *is a martingale and a non-decreasing process and is therefore null. Thus we recover the standard BSDE theory.*

Finally, we would like to emphasize that in the language of G-expectation of Peng [89], (2.2.7) is equivalent, at least if the family can be aggregated into a process K*, to saying that* $-K$ *is a G-martingale. This link has already observed in [103] where the authors proved the G-martingale representation property, which formally corresponds to a 2BSDE with a generator equal to* 0*.*

2.3 A priori estimates and uniqueness of the solution

Before proving some a priori estimates for the solution of the 2BSDE (2.2.5), we will first prove rigorously the intuition given in the Introduction 1 saying that the solution of the 2BSDE should be, in some sense, a supremum of solution of standard BSDEs. Hence, for any $\mathbb{P} \in \mathcal{P}_H$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^\infty(\mathbb{P})$, we define $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau, \xi), z^{\mathbb{P}}(\tau, \xi))$ as the unique solution of the following standard BSDE (existence and uniqueness have been proved under our assumptions by Tevzadze in [107])

$$y_t^{\mathbb{P}} = \xi - \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s,\ 0 \leqslant t \leqslant \tau,\ \mathbb{P} - a.s. \tag{2.3.1}$$

First, we introduce the following simple generalization of the comparison Theorem proved in [107] (see Theorem 2).

Proposition 2.3.1. *Let Assumptions 2.2.2 hold true. Let* ξ^1 *and* $\xi^2 \in L^\infty(\mathbb{P})$ *for some probability measure* $\mathbb{P}$*, and* V^i*,* $i = 1, 2$ *be two adapted, càdlàg nondecreasing processes null at* 0*. Let* $(Y^i, Z^i) \in \mathbb{D}^\infty(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$*,* $i = 1, 2$ *be the solutions of the following BSDE*

$$Y_t^i = \xi^i - \int_t^T \widehat{F}_s(Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dB_s + V_T^i - V_t^i,\ \mathbb{P} - a.s.,\ i = 1, 2,$$

respectively. If $\xi^1 \geqslant \xi^2$*,* $\mathbb{P} - a.s.$ *and* $V^1 - V^2$ *is nondecreasing, then it holds* $\mathbb{P} - a.s.$ *that for all* $t \in [0, T]$

$$Y_t^1 \geqslant Y_t^2.$$

Proof. First of all, we need to justify the existence of the solutions to those BSDEs. Actually, this is a simple consequence of the existence results of Tevzadze [107] and for instance Proposition 3.1 in [76]. Then, the above comparison is a mere generalization of Theorem 2 in [107]. □

We then have similarly as in Theorem 4.4 of [101] the following results which justifies the PDE intuition given in the Introduction.

Theorem 2.3.1. *Let Assumptions 2.2.2 hold. Assume $\xi \in \mathbb{L}^\infty_H$ and that $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (2.2.5). Then, for any $\mathbb{P} \in \mathcal{P}_H$ and $0 \leqslant t_1 < t_2 \leqslant T$,*

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}_H(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \; y^{\mathbb{P}'}_{t_1}(t_2, Y_{t_2}), \;\; \mathbb{P}-a.s. \tag{2.3.2}$$

Consequently, the 2BSDE (2.2.5) has at most one solution in $\mathbb{D}^\infty_H \times \mathbb{H}^2_H$.

Before proceeding with the proof, we will need the following Lemma which shows that in our 2BSDE framework, we still have a deep link between quadratic growth and the BMO spaces.

Lemma 2.3.1. *Let Assumption 2.2.2 hold. Assume $\xi \in \mathbb{L}^\infty_H$ and that $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (2.2.5). Then $Z \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$.*

Proof. Denote $\mathcal{T}_0^T$ the collection of stopping times taking values in $[0,T]$ and for each $\mathbb{P} \in \mathcal{P}_H$, let $(\tau_n^{\mathbb{P}})_{n \geqslant 1}$ be a localizing sequence for the $\mathbb{P}$-local martingale $\int_0^\cdot Z_s dB_s$. By Itô's formula under $\mathbb{P}$ applied to $e^{-\nu Y_t}$, which is a càdlàg process, for some $\nu > 0$, we have for every $\tau \in \mathcal{T}_0^T$

$$\begin{aligned}\frac{\nu^2}{2}\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_t}\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt = e^{-\nu Y_{\tau_n^{\mathbb{P}}}} - e^{-\nu Y_\tau} - \nu\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_{t^-}}dK_t^{\mathbb{P}} + \nu\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_t}\widehat{F}_t(Y_t,Z_t)dt \\ +\nu\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_{t^-}}Z_t dB_t - \sum_{\tau \leqslant s \leqslant \tau_n^{\mathbb{P}}} e^{-\nu Y_s} - e^{-\nu Y_{s^-}} + \nu \Delta Y_s e^{-\nu Y_{s^-}}.\end{aligned}$$

Since $Y \in \mathbb{D}^\infty_H$, $K^{\mathbb{P}}$ is nondecreasing and since the contribution of the jumps is negative because of the convexity of the function $x \to e^{-\nu x}$, we obtain with Assumption 2.2.1(iv)

$$\begin{aligned}\frac{\nu^2}{2}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_t}\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt\right] \leqslant e^{\nu\|Y\|_{\mathbb{D}^\infty_H}}\left(1+\nu T\left(\alpha+\beta\|Y\|_{\mathbb{D}^\infty_H}\right)\right) \\ +\frac{\nu\gamma}{2}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_t}\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt\right].\end{aligned}$$

By choosing $\nu = 2\gamma$, we then have

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^{\tau_n^{\mathbb{P}}} e^{-2\gamma Y_t}\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt\right] \leqslant \frac{1}{\gamma}e^{2\gamma\|Y\|_{\mathbb{D}^\infty_H}}\left(1+2\gamma T\left(\alpha+\beta\|Y\|_{\mathbb{D}^\infty_H}\right)\right).$$

Finally, by monotone convergence and Fatou's lemma we get that

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] \leqslant \frac{1}{\gamma} e^{4\gamma\|Y\|_{\mathbb{D}_H^\infty}} \left(1 + 2\gamma T\left(\alpha + \beta\|Y\|_{\mathbb{D}_H^\infty}\right)\right),$$

which provides the result by arbitrariness of $\mathbb{P}$ and τ. □

Proof. [Proof of Theorem 2.3.1] The proof follows the lines of the proof of Theorem 4.4 in [101], but we have to deal with some specific difficulties due to our quadratic growth assumption. First (2.3.2) implies that

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} y_t^{\mathbb{P}'}(T,\xi),\ t \in [0,T],\ \mathbb{P} - a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H,$$

and thus is unique. Then, since we have that $d\langle Y, B\rangle_t = Z_t d\langle B\rangle_t$, $\mathcal{P}_H - q.s.$, Z is also unique. We now prove (2.3.2) in three steps. Roughly speaking, we will obtain one inequality using the comparison theorem, and the other one by using the minimal condition (2.2.7).

(i) Fix $0 \leqslant t_1 < t_2 \leqslant T$ and $\mathbb{P} \in \mathcal{P}_H$. For any $\mathbb{P}' \in \mathcal{P}_H(t_1^+,\mathbb{P})$, we have

$$Y_t = Y_{t_2} - \int_t^{t_2} \widehat{F}_s(Y_s, Z_s)ds - \int_t^{t_2} Z_s dB_s + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'},\ t_1 \leqslant t \leqslant t_2,\ \mathbb{P}' - a.s.$$

and that $K^{\mathbb{P}'}$ is nondecreasing, $\mathbb{P}' - a.s.$ Then, we can apply the comparison Theorem 2.3.1 under $\mathbb{P}'$ to obtain $Y_{t_1} \geqslant y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$, $\mathbb{P}' - a.s.$ Since $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t^+$, we get $Y_{t_1} \geqslant y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$, $\mathbb{P} - a.s.$ and thus

$$Y_{t_1} \geqslant \underset{\mathbb{P}' \in \mathcal{P}_H(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_H$. Let us assume for the moment that

$$C_{t_1}^{\mathbb{P},p} := \underset{\mathbb{P}' \in \mathcal{P}_H(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^p\right] < +\infty,\ \mathbb{P} - a.s., \text{ for all } p \geqslant 1.$$

For every $\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})$, denote

$$\delta Y := Y - y^{\mathbb{P}'}(t_2, Y_{t_2}) \text{ and } \delta Z := Z - z^{\mathbb{P}'}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption 2.2.2(vi) and the local Lipschitz Assumption 2.2.2(v), there exist a bounded process λ and a process η with

$$|\eta_t| \leqslant \mu\left(\left|\widehat{a}_t^{1/2} Z_t\right| + \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P}'}\right|\right),\ \mathbb{P}' - a.s.$$

such that

$$\delta Y_t = \int_t^{t_2} \left(\lambda_s \delta Y_s + (\eta_s + \phi_s)\widehat{a}_s^{1/2}\delta Z_s\right) ds - \int_t^{t_2} \delta Z_s dB_s + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'},\ t \leqslant t_2,\ \mathbb{P}' - a.s.$$

Define for $t_1 \leqslant t \leqslant t_2$

$$M_t := \exp\left(\int_{t_1}^t \lambda_s ds\right),\ \mathbb{P}^{'} - a.s.$$

Now, since $\phi \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$, by Lemma 2.3.1, we know that the $\mathbb{P}^{'}$-exponential martingale $\mathcal{E}\left(\int_0^{\cdot}(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)$ is a $\mathbb{P}^{'}$-uniformly integrable martingale (see Theorem 2.3 in the book by Kazamaki [59]). Therefore we can define a probability measure $\mathbb{Q}^{'}$, which is equivalent to $\mathbb{P}^{'}$, by its Radon-Nykodym derivative

$$\frac{d\mathbb{Q}^{'}}{d\mathbb{P}^{'}} = \mathcal{E}\left(\int_0^T(\phi_s+\eta_s))\widehat{a}_s^{-1/2}dB_s\right).$$

Then, by Itô's formula, we obtain, as in [101], that

$$\delta Y_{t_1} = \mathbb{E}_{t_1}^{\mathbb{Q}^{'}}\left[\int_{t_1}^{t_2} M_t dK_t^{\mathbb{P}^{'}}\right] \leqslant \mathbb{E}_{t_1}^{\mathbb{Q}^{'}}\left[\sup_{t_1\leqslant t\leqslant t_2}(M_t)(K_{t_2}^{\mathbb{P}^{'}}-K_{t_1}^{\mathbb{P}^{'}})\right],$$

since $K^{\mathbb{P}^{'}}$ is nondecreasing. Then, since λ is bounded, we have that M is also bounded and thus for every $p \geqslant 1$

$$\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\sup_{t_1\leqslant t\leqslant t_2}(M_t)^p\right] \leqslant C_p,\ \mathbb{P}^{'} - a.s. \tag{2.3.3}$$

Since $(\eta+\phi)\widehat{a}_s^{-1/2}$ is in $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$, we know by Lemma 2.2.1 that there exists $r > 1$, independent of the probability measure considered, such that $\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right) \in L_H^r$. Then it follows from the Hölder inequality and Bayes Theorem that

$$\begin{aligned}\delta Y_{t_1} &\leqslant \frac{\left(\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\mathcal{E}\left(\int_0^{t_2}(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)^r\right]\right)^{\frac{1}{r}}}{\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\mathcal{E}\left(\int_0^{t_2}(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\right]}\left(\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\left(\sup_{t_1\leqslant t\leqslant t_2} M_t\right)^q\left(K_{t_2}^{\mathbb{P}^{'}}-K_{t_1}^{\mathbb{P}^{'}}\right)^q\right]\right)^{\frac{1}{q}}\\ &\leqslant C\left(C_{t_1}^{\mathbb{P},4q-1}\right)^{\frac{1}{4q}}\left(\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[K_{t_2}^{\mathbb{P}^{'}}-K_{t_1}^{\mathbb{P}^{'}}\right]\right)^{\frac{1}{4q}}.\end{aligned}$$

By the minimum condition (2.2.7) and since $\mathbb{P}^{'} \in \mathcal{P}_H(t^+,\mathbb{P})$ is arbitrary, this ends the proof.

(iii) It remains to show that the estimate for $C_{t_1}^{\mathbb{P},p}$ holds for $p \geqslant 1$. By definition of the family $\left\{K^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_H\right\}$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{'}}\left[\left(K_{t_2}^{\mathbb{P}^{'}}-K_{t_1}^{\mathbb{P}^{'}}\right)^p\right] &\leqslant C\left(1+\|Y\|_{\mathbb{D}_H^\infty}^p+\|\xi\|_{\mathbb{L}_H^\infty}^p+\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\left(\int_{t_1}^{t_2}\left|\widehat{a}_t^{1/2}Z_t\right|^2dt\right)^p\right]\right)\\ &\quad + C\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\left(\int_{t_1}^{t_2} Z_t dB_t\right)^p\right].\end{aligned}$$

Thus by the energy inequalities for BMO martingales and by Burkholder-Davis-Gundy inequality, we get that

$$\mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}\right)^p\right]\leqslant C\left(1+\|Y\|_{\mathbb{D}_H^\infty}^p+\|\xi\|_{\mathbb{L}_H^\infty}^p+\|Z\|_{\mathbb{BMO}_\mathrm{H}}^{2p}+\|Z\|_{\mathbb{BMO}_\mathrm{H}}^p\right).$$

Therefore, we have proved that

$$\sup_{\mathbb{P}'\in\mathcal{P}_H(t_1^+,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}\right)^p\right]<+\infty.$$

Then we proceed exactly as in the proof of Theorem 4.4 in [101]. □

Remark 2.3.1. *It is interesting to notice that in contrast with standard quadratic BSDEs, for which the only property of BMO martingales used to obtain uniqueness is the fact that their Doléans-Dade exponential is a uniformly integrable martingale, we need a lot more in the 2BSDE framework. Indeed, we use extensively the energy inequalities and the existence of moments for the Doléans-Dade exponential (which is a consequence of the so called reverse Hölder inequalities, which is a more general version of Lemma 2.2.1). Furthermore, we will also use the so-called Muckenhoupt condition (which corresponds to Lemma 2.2.2, see [59] for more details) in both our proofs of existence. This seems to be directly linked to the presence of the non-decreasing processes $K^{\mathbb{P}}$ and raises the question about whether it could be possible to generalize the recent approach of Barrieu and El Karoui [6], to second-order BSDEs. Indeed, since they no longer assume a bounded terminal condition, the Z part of the solution is no-longer BMO. We leave this interesting but difficult question to future research.*

We conclude this section by showing some *a priori* estimates which will be useful in the sequel. Notice that these estimates also imply uniqueness, but they use intensively the representation formula (2.3.2).

Theorem 2.3.2. *Let Assumption 2.2.2 hold.*

(i) *Assume that $\xi\in\mathbb{L}_H^\infty$ and that $(Y,Z)\in\mathbb{D}_H^\infty\times\mathbb{H}_H^2$ is a solution to 2BSDE (2.2.5). Then, there exists a constant C such that*

$$\|Y\|_{\mathbb{D}_H^\infty}+\|Z\|_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})}^2\leqslant C\left(1+\|\xi\|_{\mathbb{L}_H^\infty}\right)$$
$$\forall p\geqslant 1,\quad\sup_{\mathbb{P}\in\mathcal{P}_H,\,\tau\in\mathcal{T}_0^T}\mathbb{E}_\tau^{\mathbb{P}}\left[(K_T^{\mathbb{P}}-K_\tau^{\mathbb{P}})^p\right]\leqslant C\left(1+\|\xi\|_{\mathbb{L}_H^\infty}^p\right).$$

(ii) *Assume that $\xi^i\in\mathbb{L}_H^\infty$ and that $(Y^i,Z^i)\in\mathbb{D}_H^\infty\times\mathbb{H}_H^2$ is a corresponding solution to 2BSDE (2.2.5), $i=1,2$. Denote $\delta\xi:=\xi^1-\xi^2$, $\delta Y:=Y^1-Y^2$, $\delta Z:=Z^1-Z^2$ and $\delta K^{\mathbb{P}}:=K^{\mathbb{P},1}-K^{\mathbb{P},2}$. Then, there exists a constant C such that*

$$\|\delta Y\|_{\mathbb{D}_H^\infty}\leqslant C\|\delta\xi\|_{\mathbb{L}_H^\infty}$$
$$\|\delta Z\|_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})}^2\leqslant C\|\delta\xi\|_{\mathbb{L}_H^\infty}\left(1+\left\|\xi^1\right\|_{\mathbb{L}_H^\infty}+\left\|\xi^2\right\|_{\mathbb{L}_H^\infty}\right)$$
$$\forall p\geqslant 1,\ \sup_{\mathbb{P}\in\mathcal{P}_H}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}\left|\delta K_t^{\mathbb{P}}\right|^p\right]\leqslant C\|\xi\|_{\mathbb{L}_H^\infty}^{\frac{p}{2}}\left(1+\left\|\xi^1\right\|_{\mathbb{L}_H^\infty}^{\frac{p}{2}}+\left\|\xi^2\right\|_{\mathbb{L}_H^\infty}^{\frac{p}{2}}\right).$$

Proof.

(i) By Theorem 2.3.1 we know that for all $\mathbb{P} \in \mathcal{P}_H$ and for all $t \in [0,T]$ we have

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}} y_t^{\mathbb{P}'} \ \mathbb{P}-a.s..$$

Then by Lemma 1 in [12], we know that for all $\mathbb{P} \in \mathcal{P}_H$

$$\left|y_t^{\mathbb{P}}\right| \leqslant \frac{1}{\gamma}\log\left(\mathbb{E}_t^{\mathbb{P}}\left[\psi(|\xi|)\right]\right), \text{ where } \psi(x) := \exp\left(\gamma\alpha\frac{e^{\beta T}-1}{\beta} + \gamma e^{\beta T}x\right).$$

Thus, we obtain

$$\left|y_t^{\mathbb{P}}\right| \leqslant \alpha\frac{e^{\beta T}-1}{\beta} + e^{\beta T}\left\|\xi\right\|_{\mathbb{L}_H^\infty},$$

and by the representation recalled above, the estimate of $\|Y\|_{\mathbb{D}_H^\infty}$ is obvious.

By the proof of Lemma 2.3.1, we have now

$$\|Z\|^2_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})} \leqslant Ce^{C\|Y\|_{\mathbb{D}_H^\infty}}\left(1+\|Y\|_{\mathbb{D}_H^\infty}\right) \leqslant C\left(1+\|\xi\|_{\mathbb{L}_H^\infty}\right).$$

Finally, we have for all $\tau \in \mathcal{T}_0^T$, for all $\mathbb{P} \in \mathcal{P}_H$ and for all $p \geqslant 1$, by definition

$$(K_T^{\mathbb{P}} - K_\tau^{\mathbb{P}})^p = \left(Y_\tau - \xi + \int_\tau^T \widehat{F}_t(Y_y, Z_t)dt + \int_\tau^T Z_t dB_t\right)^p.$$

Therefore, by our growth Assumption 2.2.1(iv)

$$\begin{aligned}
\mathbb{E}_\tau^{\mathbb{P}}\left[(K_T^{\mathbb{P}} - K_\tau^{\mathbb{P}})^p\right] &\leqslant C\left(1 + \|\xi\|^p_{\mathbb{L}_H^\infty} + \|Y\|^p_{\mathbb{D}_H^\infty} + \mathbb{E}_\tau^{\mathbb{P}}\left[\left(\int_\tau^T \left|\widehat{a}_t^{1/2}Z_t\right|^2 dt\right)^p\right]\right) \\
&\quad + C\mathbb{E}_\tau^{\mathbb{P}}\left[\left(\int_\tau^T Z_t dB_t\right)^p\right] \\
&\leqslant C\left(1 + \|\xi\|^p_{\mathbb{L}_H^\infty} + \|Z\|^{2p}_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})} + \|Z\|^p_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})}\right) \\
&\leqslant C\left(1 + \|\xi\|^p_{\mathbb{L}_H^\infty}\right),
\end{aligned}$$

where we used again the energy inequalities and the BDG inequality. This provides the estimate for $K^{\mathbb{P}}$ by arbitrariness of τ and $\mathbb{P}$.

(ii) With the same notations and calculations as in step (ii) of the proof of Theorem 2.3.1, it is easy to see that for all $\mathbb{P} \in \mathcal{P}_H$ and for all $t \in [0,T]$, we have

$$\delta y_t^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{Q}}\left[M_T\delta\xi\right] \leqslant C\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty},$$

since M is bounded and we have (2.3.3). By Theorem 2.3.1, the estimate for δY follows.

Now apply Itô's formula under a fixed $\mathbb{P} \in \mathcal{P}_H$ to $|\delta Y|^2$ between a given stopping time $\tau \in \mathcal{T}_0^T$ and T

$$\mathbb{E}_\tau^{\mathbb{P}}\left[|\delta Y_\tau|^2 + \int_\tau^T \left|\widehat{a}_t^{1/2}\delta Z_t\right|^2 dt\right] \leqslant \mathbb{E}_\tau^{\mathbb{P}}\left[|\delta\xi|^2 - 2\int_\tau^T \delta Y_t\left(\widehat{F}_t(Y_t^1, Z_t^1) - \widehat{F}_t(Y_t^2, Z_t^2)\right) dt\right] + 2\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T \delta Y_{t-} d(\delta K_t^{\mathbb{P}})\right].$$

Then, we have by Assumption 2.2.1(iv) and the estimates proved in (i) above

$$\begin{aligned}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T \left|\widehat{a}_t^{1/2}\delta Z_t\right|^2 dt\right] &\leqslant C\left\|\delta Y\right\|_{\mathbb{D}_H^\infty}\left(1 + \sum_{i=1}^2 \left\|Y^i\right\|_{\mathbb{D}_H^\infty} + \left\|Z^i\right\|_{\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})}\right) \\ &\quad + \left\|\delta\xi\right\|_{\mathbb{L}_H^\infty}^2 + 2\left\|\delta Y\right\|_{\mathbb{D}_H^\infty}\mathbb{E}_\tau^{\mathbb{P}}\left[\left|K_T^{\mathbb{P},1} - K_\tau^{\mathbb{P},1}\right| + \left|K_T^{\mathbb{P},2} - K_\tau^{\mathbb{P},2}\right|\right] \\ &\leqslant C\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty}\left(1 + \left\|\xi^1\right\|_{\mathbb{L}_H^\infty} + \left\|\xi^2\right\|_{\mathbb{L}_H^\infty}\right),\end{aligned}$$

which implies the required estimate for δZ.

Finally, by definition, we have for all $\mathbb{P} \in \mathcal{P}_H$ and for all $t \in [0, T]$

$$\delta K_t^{\mathbb{P}} = \delta Y_0 - \delta Y_t + \int_0^t \widehat{F}_s(Y_s^1, Z_s^1) - \widehat{F}_s(Y_s^2, Z_s^2) ds + \int_0^t \delta Z_s dB_s.$$

By Assumptions 2.2.2(iv) and (vi), it follows that

$$\begin{aligned}\sup_{0 \leqslant t \leqslant T}\left|\delta K_t^{\mathbb{P}}\right| &\leqslant C\left(\left\|\delta Y\right\|_{\mathbb{D}_H^\infty} + \int_0^T \left|\widehat{a}_s^{1/2}\delta Z_s\right|\left(1 + \left|\widehat{a}_s^{1/2}Z_s^1\right| + \left|\widehat{a}_s^{1/2}Z_s^2\right|\right) ds\right) \\ &\quad + \sup_{0 \leqslant t \leqslant T}\left|\int_0^t \delta Z_s dB_s\right|,\end{aligned}$$

and by Cauchy-Schwarz, BDG and energy inequalities, we see that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leqslant t \leqslant T}\left|\delta K_t^{\mathbb{P}}\right|^p\right] &\leqslant C\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left(1 + \left|\widehat{a}_s^{1/2}Z_s^1\right|^2 + \left|\widehat{a}_s^{1/2}Z_s^2\right|^2\right) ds\right)^p\right]^{\frac{1}{2}} \\ &\quad \times \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{a}_s^{1/2}\delta Z_s\right|^2 ds\right)^p\right]^{\frac{1}{2}} \\ &\quad + C\left(\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty}^p + \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{a}_s^{1/2}\delta Z_s\right|^2 ds\right)^{p/2}\right]\right) \\ &\leqslant C\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty}^{p/2}\left(1 + \left\|\xi^1\right\|_{\mathbb{L}_H^\infty}^{p/2} + \left\|\xi^2\right\|_{\mathbb{L}_H^\infty}^{p/2}\right).\end{aligned}$$

□

Remark 2.3.2. *Let us note that the proof of* (i) *only requires that Assumption 2.2.2*(iv) *holds true, whereas* (ii) *also requires Assumption 2.2.2*(v) *and* (vi).

2.4 2BSDEs and monotone approximations

This Section is devoted to the study of monotone approximations in the 2BSDE framework. We start with the simplest quadratic 2BSDEs, which allows us to introduce a quasi-sure version of the entropic risk measure. In that case, we obtain existence through the classical exponential change. Then, we show that for more general generators, this approach usually fails because of the absence of a general quasi-sure monotone convergence Theorem. Finally, we prove an existence result using another type of approximation which has the property to be stationary.

2.4.1 Entropy and purely quadratic 2BSDEs

Given $\xi \in \mathcal{L}_H^\infty$, we first consider the purely quadratic 2BSDE defined as follows

$$Y_t = -\xi + \int_t^T \frac{\gamma}{2}\left|\widehat{a}_s^{1/2}Z_s\right|^2 ds - \int_t^T Z_s dB_s + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ 0 \leqslant t \leqslant T,\ \mathcal{P}_H - q.s. \tag{2.4.1}$$

Then we use the classical exponential change of variables and define

$$\overline{Y}_t := e^{\gamma Y_t},\ \overline{Z}_t := \gamma \overline{Y}_t Z_t,\ \overline{K}_t^{\mathbb{P}} := \gamma \int_0^t \overline{Y}_s dK_s^{\mathbb{P}} - \sum_{0 \leqslant s \leqslant t} e^{\gamma Y_s} - e^{\gamma Y_{s^-}} - \gamma \Delta Y_s e^{\gamma Y_{s^-}}.$$

At least formally, we see that $(\overline{Y}, \overline{Z}, \overline{K}^{\mathbb{P}})$ verifies the following equation

$$\overline{Y}_t = e^{-\gamma\xi} - \int_t^T \overline{Z}_s dB_s + \overline{K}_T^{\mathbb{P}} - \overline{K}_t^{\mathbb{P}},\ 0 \leqslant t \leqslant T,\ \mathbb{P} - a.s.\ \forall \mathbb{P} \in \mathcal{P}_H \tag{2.4.2}$$

which is in fact a 2BSDE with generator equal to 0 (and thus Lipschitz), provided that the familly $\left(\overline{K}^{\mathbb{P}}\right)_{\mathbb{P}\in\mathcal{P}_H}$ satisfies the minimum condition (2.2.7). Thus the purely quadratic 2BSDE (2.4.1) is linked to the 2BSDE with Lipschitz generator (2.4.2), which has a unique solution by Soner, Touzi and Zhang [101]. We now make this rigorous.

Proposition 2.4.1. *The 2BSDE (2.4.1) has a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ given by*

$$Y_t = \frac{1}{\gamma} \ln\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}\left[e^{-\gamma\xi}\right]\right),\ \mathbb{P} - a.s.,\ t \in [0,T],\ \textit{for all}\ \mathbb{P} \in \mathcal{P}_H.$$

Proof. Uniqueness is a simple consequence of Theorem 2.3.1. In the following, we prove the existence in 3 steps.
Step 1: Let $(\overline{Y}, \overline{Z}) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ be the unique solutinon to the 2BSDE (2.4.2) and $\overline{K}^{\mathbb{P}}$ be the correponding non-decreasing processes. In particular, we know that

$$\overline{Y}_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}\left[e^{-\gamma\xi}\right],\ \mathbb{P} - a.s.,$$

which implies that $\overline{Y} \in \mathbb{D}_H^\infty$, since

$$0 < e^{-\gamma\|\xi\|_{\mathbb{L}_H^\infty}} \leqslant Y_t \leqslant e^{\gamma\|\xi\|_{\mathbb{L}_H^\infty}}.$$

We can therefore make the following change of variables

$$Y_t := \frac{1}{\gamma}\ln\left(\overline{Y}_t\right),\ Z_t := \frac{1}{\gamma}\frac{\overline{Z}_t}{\overline{Y}_t}.$$

Then by Itô's formula, we can verify that the pair $(Y,Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ satisfies (2.4.1) with

$$K_t^{\mathbb{P}} := \int_0^t \frac{1}{\gamma \overline{Y}_s} d\overline{K}_s^{\mathbb{P},c} - \sum_{0<s\leqslant t} \frac{1}{\gamma}\log\left(1 - \frac{\Delta \overline{K}_s^{\mathbb{P},d}}{\overline{Y}_{s-}}\right).$$

Moreover, notice that $K^{\mathbb{P}}$ is non-decreasing with $K_0^{\mathbb{P}} = 0$.

Step 2: Denote now $(y^{\mathbb{P}}, z^{\mathbb{P}})$ the solutions of the standard BSDEs corresponding to the 2BSDE (2.4.1) (existence and uniqueness are ensured for example by [107]). Furthermore, if we define

$$\overline{y}_t^{\mathbb{P}} := e^{\gamma y_t^{\mathbb{P}}},\ \overline{z}_t^{\mathbb{P}} := \gamma \overline{y}_t^{\mathbb{P}} z_t^{\mathbb{P}},$$

then we know that $(\overline{y}^{\mathbb{P}}, \overline{z}^{\mathbb{P}})$ solve the standard BSDE under $\mathbb{P}$ corresponding to (2.4.2). Due to the monotonicity of the function $x \rightarrow \ln(x)$ and the representation for $\overline{Y}$

$$\overline{Y}_t = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \overline{y}_t^{\mathbb{P}} = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[e^{-\gamma\xi}\right],\ \mathbb{P}-a.s.,$$

we have the following representation for Y

$$Y_t = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_t^{\mathbb{P}} = \frac{1}{\gamma}\ln\left(\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[e^{-\gamma\xi}\right]\right),\ \mathbb{P}-a.s.$$

Step 3: Finally, it remains to check the minimum condition for the family of non-decreasing processes $\{K^{\mathbb{P}}\}$. Since the purely quadratic generator satisfies the Assumption 2.2.1, we can derive the minimum condition from the above representation for Y exactly as in the proof of Theorem 2.4.1 in Subsection 2.4.3. □

Thanks to the above result, we can define a quasi-sure (or robust) version of the entropic risk measure under volatility uncertainty

$$e_{\gamma,t}(\xi) := \frac{1}{\gamma}\ln\left(\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[e^{-\gamma\xi}\right]\right),$$

where the parameter γ stands for the risk tolerance. We emphasize that, as proved in [102] (see Proposition 4.11), the solution of (2.4.1) is actually $\mathbb{F}$-measurable, so we also have

$$e_{\gamma,t}(\xi) := \frac{1}{\gamma}\ln\left(\underset{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\mathbb{E}_t^{\mathbb{P}'}\left[e^{-\gamma\xi}\right]\right),$$

which in particular implies that

$$e_{\gamma,0}(\xi) = \frac{1}{\gamma}\ln\left(\sup_{\mathbb{P}\in\mathcal{P}_H}\mathbb{E}^{\mathbb{P}}\left[e^{-\gamma\xi}\right]\right).$$

More generally, by the same exponential change and arguments above, we can also prove that there exists a unique solution to 2BSDEs with terminal condition $\xi \in \mathcal{L}_H^\infty$ and the following type of quadratic growth generators $\widehat{a}_t^{1/2} z g(t,\omega) + h(t,\omega) - \frac{\gamma}{2}\left|\widehat{a}_t^{1/2} z\right|^2$ where g and h are assumed to be bounded, adapted and uniformly continuous in ω for the $\|\cdot\|_\infty$.

2.4.2 Why the exponential transformation may fail in general?

Coming back to Kobylanski [63], we know that the exponential transformation used in the previous subsection is an important tool in the study of quadratic BSDEs. However, unlike with a purely quadratic generator, in the general case the exponential change does not lead immediately to a Lipschitz BSDE. For the sake of clarity, let us consider the 2BSDE (2.2.5) and let us denote

$$\eta := e^{\gamma\xi},\ \overline{Y}_t := e^{\gamma Y_t},\ \overline{Z}_t := \gamma\overline{Y}_t Z_t,\ \overline{K}_t^{\mathbb{P}} := \gamma\int_0^t \overline{Y}_s dK_s^{\mathbb{P}} - \sum_{0\leqslant s\leqslant t} e^{\gamma Y_s} - e^{\gamma Y_{s^-}} - \gamma\Delta Y_s e^{\gamma Y_{s^-}}.$$

Then we expect that, at least formally, if (Y,Z) is a solution of (2.2.5), then $(\overline{Y},\overline{Z})$ is a solution of the following 2BSDE

$$\overline{Y}_t = \eta - \gamma\int_t^T \overline{Y}_s\left(\widehat{F}_s\left(\frac{\log\overline{Y}_s}{\gamma},\frac{\overline{Z}_s}{\gamma\overline{Y}_s}\right)+\frac{\left|\widehat{a}_s^{1/2}\overline{Z}_s\right|^2}{2\gamma\overline{Y}_s^2}\right)ds - \int_t^T \overline{Z}_s dB_s + \overline{K}_T^{\mathbb{P}} - \overline{K}_t^{\mathbb{P}}. \quad (2.4.3)$$

Let us now define for $(t,y,z)\in[0,T]\times\mathbb{R}_+^*\times\mathbb{R}^d$,

$$G_t(\omega,y,z) := \gamma y\left(\widehat{F}_t\left(\omega,\frac{\log y}{\gamma},\frac{z}{\gamma y}\right)+\frac{\left|\widehat{a}_t^{1/2}z\right|^2}{2\gamma y^2}\right).$$

Then, despite the fact that the generator G is not Lipschitz, it is possible, as shown by Kobylanski [63], to find a sequence $(G^n)_{n\geqslant 0}$ of Lipschitz functions which decreases to G. Then, it is possible, thanks to the result of [101] to define for each n the solution (Y^n,Z^n) of the corresponding 2BSDE. The idea is then to prove existence and uniqueness of a solution for the 2BSDE with generator G (and thus also for the 2BSDE (2.2.5)) by passing to the limit in some sense in the sequence (Y^n,Z^n).

If we then follow the usual approach for standard BSDEs, the first step is to argue that thanks to the comparison theorem (which still holds true for Lipschitz 2BSDEs, see [101]), the sequence Y^n is decreasing, and thanks to a priori estimates that it must converge $\mathcal{P}_H - q.s.$ to some process Y. And this is exactly now that the situation becomes much more complicated with 2BSDEs. Indeed, if we were in the classical framework, this convergence of Y^n together with the a priori estimates would be sufficient to prove the convergence in the usual $\mathbb{H}^2$ space, thanks to the dominated convergence theorem. However, in our case, since the norms involve the supremum over a family of probability measures, this theorem can fail (we refer the reader to Section 2.6 in [90] for more details). Therefore, we cannot obtain directly that

$$\sup_{\mathbb{P}\in\mathcal{P}_H}\mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_t^n - Y_t|^2\,dt\right]\underset{n\to+\infty}{\longrightarrow} 0,$$

which is a crucial step in the approximation proof.

This is precisely the major difficulty when considering the 2BSDE framework. The only monotone convergence Theorem in a similar setting has been proved by Denis, Hu and

Peng (see [28]). However, one need to consider random variables X^n which are regular in ω, more precisely quasi-continuous, that is to say that for every $\varepsilon > 0$, there exists an open set $\mathcal{O}^\varepsilon$ such that the X^n are continuous in ω outside $\mathcal{O}^\varepsilon$ and such that

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{P}(\mathcal{O}^\varepsilon) \leqslant \varepsilon.$$

Moreover, the set of probability measures considered must be weakly compact. This induces several fundamental problems when one tries to apply directly this Theorem to $(Y^n)_{n\geqslant 0}$.

(i) *First, if we assume that the terminal condition ξ is in $UC_b(\Omega)$, since the generator $\widehat{F}$ (and thus G^n) are uniformly continuous in ω, we can reasonably expect to be able to prove that the Y^n will be also continuous in ω, $\mathbb{P}-a.s.$, for every $\mathbb{P}\in\mathcal{P}_H$. However, this is clearly not sufficient to obtain the quasi-continuity. Indeed, for each $\mathbb{P}$, we would have a $\mathbb{P}$-negligible set outside of which the Y^n are continuous in ω. But since the probability measures are mutually singular, this does not imply the existence of the open set of the definition of quasi-continuity.*

We moreover emphasize that it is a priori a very difficult problem to show the quasi-continuity of the solution of a 2BSDE, because by definition, it is defined $\mathbb{P}-a.s.$ for every $\mathbb{P}$, and the quasi-continuity is by essence a notion related to the theory of capacities, not of probability measures.

(ii) *Next, it has been shown that if we assume that the matrices $\underline{a}^{\mathbb{P}}$ and $\overline{a}^{\mathbb{P}}$ appearing in Definition 2.2.1 are uniform in $\mathbb{P}$, then the set $\mathcal{P}_H$ is only weakly relatively compact. Then, we are left with two options. First, we can restrict ourselves to a closed subset of $\mathcal{P}_H$, which will therefore be weakly compact. However, as pointed out in [102], it is not possible to restrict arbitrarily the probability measures considered. Indeed, since the whole approach of [101] to prove existence of Lipschitz 2BSDEs relies on stochastic control and the dynamic programming equation, we need the set of processes α in the definition of $\overline{\mathcal{P}}_S$ (that is to say our set of control processes) to be stable by concatenation and bifurcation (see for instance Remark* 3.1 *in [17]) in order to recover the results of [101]. And it is not clear at all to us whether it is possible to find a closed subset of $\mathcal{P}_H$ satisfying this stability properties.*

Otherwise, we could work with the weak closure of $\mathcal{P}_H$. The problem now is that the probability measures in that closure no longer satisfy necessarily the martingale representation property and the 0-1 Blumenthal law. In that case (since the filtration $\mathbb{F}$ will only be quasi-left continuous), and as already shown by El Karoui and Huang [32], we would need to redefine a solution of a 2BSDE by adding a martingale orthogonal to the canonical process. However, defining such solutions is a complicated problem outside of the scope of this paper.

We hope to have convinced the reader that because of all the reasons listed above, it seems difficult in general to prove existence of a solution to a 2BSDE using approximation arguments. However, the situation is not hopeless. Indeed, in [90], the author uses such an approach to prove existence of a solution to a 2BSDE with a generator with linear growth satisfying some monotonicity condition. The idea is that in this case it is possible to show that the sequence of approximated generators converges uniformly in (y, z), and this allows to have a control on the difference $|Y_t^n - Y_t|$ by a quantity which is regular enough to apply the monotone convergene Theorem of [28]. Nonetheless, this relies heavily on the type of approximation used and cannot a priori be extended to more general cases.

Notwithstanding this, we will show an existence result in the next subsection using an approximation which has the particularity of being stationary, which immediately solves the convergence problems that we mentioned above. This approach is based on very recent results of Briand and Elie [14] on standard quadratic BSDEs.

2.4.3 A stationary approximation

For technical reasons that we will explain below, we will work throughout this subsection under a subset of $\mathcal{P}_H$, which was first introduced in [103]. Namely, we will denote by Ξ the set of processes α satisfying

$$\alpha_t(\omega) = \sum_{n=0}^{+\infty}\sum_{i=1}^{+\infty} \alpha_t^{n,i} \mathbf{1}_{E_n^i}(\omega)\mathbf{1}_{[\tau_n(\omega),\tau_{n+1}(\omega))}(t),$$

where for each i and for each n, $\alpha^{n,i}$ is a bounded deterministic mapping, τ_n is an $\mathcal{F}$-stopping time with $\tau_0 = 0$, such that $\tau_n < \tau_{n+1}$ on $\{\tau_n < +\infty\}$, $\inf\{n \geqslant 0,\ \tau_n = +\infty\} < +\infty$, τ_n takes countably many values in some fixed $I_0 \subset [0, T]$ which is countable and dense in $[0, T]$ and for each n, $(E_i^n)_{i \geqslant 1} \subset \mathcal{F}_{\tau_n}$ forms a partition of Ω.

We will then consider the set $\widetilde{\mathcal{P}}_H := \{\mathbb{P}^\alpha \in \mathcal{P}_H,\ \alpha \in \Xi\}$. As shown in [102], this set satisfies the right stability properties (already mentioned in the previous subsection) so much so that the Lipschitz theory of 2BSDEs still holds when we are working $\widetilde{\mathcal{P}}_H - q.s.$ Notice that for the sake of simplicity, we will keep the same notations for the spaces considered under $\widetilde{\mathcal{P}}_H$ or $\mathcal{P}_H$. Let us now describe the Assumptions under which we will be working

Assumption 2.4.1. *Let Assumption 2.2.2 holds, with the addition that the process ϕ in* (v) *is bounded and that the mapping F is deterministic.*

The main result of this Section is then

Theorem 2.4.1. *Let Assumption 2.4.1 hold. Assume further that $\xi \in \mathcal{L}_H^\infty$, that it is Malliavin differentiable $\widetilde{\mathcal{P}}_H - q.s.$ and that its Malliavin derivative is in $\mathbb{D}_H^\infty$. Then the 2BSDE (2.2.5) (considered $\widetilde{\mathcal{P}}_H - q.s.$) has a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$. Moreover, the family $\{K^\mathbb{P},\ \mathbb{P} \in \widetilde{\mathcal{P}}_H\}$ can be aggregated.*

Proof. Uniqueness follows from Theorem 2.3.1, so we concentrate on the existence part. Let us define the following sequence of generators

$$F_t^n(y, z, a) := F_t\left(y, \frac{|z| \wedge n}{|z|}z, a\right), \text{ and } \widehat{F}_t^n(y, z) := F_t^n(y, z, \widehat{a}_t).$$

Then for each n, F^n is uniformly Lipschitz in (y, z) and thanks to Assumption 2.4.1, we can apply the result of [101] to obtain the existence of a solution (Y^n, Z^n) to the 2BSDE

$$Y_t^n = \xi - \int_t^T \widehat{F}_s^n(Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s + K_T^{\mathbb{P},n} - K_t^{\mathbb{P},n},\ \mathbb{P}-a.s., \text{ for all } \mathbb{P} \in \widetilde{\mathcal{P}}_H. \quad (2.4.4)$$

Moreover, we have for all $\mathbb{P} \in \widetilde{\mathcal{P}}_H$ and for all $t \in [0,T]$

$$Y_t^n = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+,\mathbb{P})} y_t^{\mathbb{P},n}, \mathbb{P} - a.s., \tag{2.4.5}$$

where $(y^{\mathbb{P},n}, z^{\mathbb{P},n})$ is the unique solution of the Lipschitz BSDE with generator $\widehat{F}^n$ and terminal condition ξ under $\mathbb{P}$. Now, using Lemma 2.1 in [14] and its proof (see Remark 2.4.1 below) under each $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, we know that the sequence $y^{\mathbb{P},n}$ is actually stationary. Therefore, by (2.4.5), this also implies that the sequence Y^n is stationary. Hence, we immediately have that Y^n converges to some Y in $\mathbb{D}_H^\infty$. Moreover, we still have the representation

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+,\mathbb{P})} y_t^{\mathbb{P}}, \mathbb{P} - a.s., \tag{2.4.6}$$

Now, identifying the martingale parts in (2.4.4), we also obtain that the sequence Z^n is stationary and thus converges trivially in $\mathbb{H}_H^2$ to some Z. For n large enough, we thus have

$$\widehat{F}_t^n(Y_t^n, Z_t^n) = \widehat{F}_t^n(Y_t, Z_t).$$

Besides, we have by Assumption 2.4.1

$$\left|\widehat{F}_t^n(Y_t, Z_t)\right| \leqslant \alpha + \beta\left|Y_t\right| + \frac{\gamma}{2}\left|\widehat{a}^{1/2}\frac{|Z_t| \wedge n}{|Z_t|}Z_t\right|^2 \leqslant \alpha + \beta\left|Y_t\right| + \frac{\gamma}{2}\left|\widehat{a}^{1/2}Z_t\right|^2, \ \widetilde{\mathcal{P}}_H - q.s.$$

Since $(Y,Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$, we can apply the dominated convergence theorem for the Lebesgue measure to obtain by continuity of F that

$$\int_0^T \widehat{F}_s^n(Y_s^n, Z_s^n)ds \underset{n\to+\infty}{\longrightarrow} \int_0^T \widehat{F}_s(Y_s, Z_s)ds, \widetilde{\mathcal{P}}_H - q.s.$$

Using this result in (2.4.4), this implies necessarily that for each $\mathbb{P}$, $K^{\mathbb{P},n}$ converges $\mathbb{P}-a.s.$ to a non-decreasing process $K^{\mathbb{P}}$. Now, in order to verify that we indeed have obtained the solution, we need to check if the processes $K^{\mathbb{P}}$ satisfy the minimum condition (2.2.7). Let $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, $t \in [0,T]$ and $\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+,\mathbb{P})$. From the proof of Theorem 2.3.1, we have with the same notations

$$\begin{aligned}\delta Y_t = \mathbb{E}_t^{\mathbb{Q}'}\left[\int_t^T M_t dK_t^{\mathbb{P}'}\right] &\geqslant \mathbb{E}_t^{\mathbb{Q}'}\left[\inf_{t\leqslant s\leqslant T}(M_s)(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})\right] \\ &= \frac{\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\inf_{t\leqslant s\leqslant T}(M_s)(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})\right]}{\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\right]}\end{aligned}$$

For notational convenience, denote $\mathcal{E}_t := \mathcal{E}\left(\int_0^t(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)$. Let r be the number given by Lemma 2.2.2 applied to $\mathcal{E}$. Then we estimate

$$\begin{aligned}
&\mathbb{E}_t\left[K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right]\\
&\leqslant \mathbb{E}_t^{\mathbb{P}'}\left[\frac{\mathcal{E}_T}{\mathcal{E}_t}\inf_{t\leqslant s\leqslant T}(M_s)(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'})\right]^{\frac{1}{2r-1}}\mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}_T}{\mathcal{E}_t}\inf_{t\leqslant s\leqslant T}(M_s)^{-1}\right)^{\frac{1}{2(r-1)}}(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'})\right]^{\frac{2(r-1)}{2r-1}}\\
&\leqslant (\delta Y_t)^{\frac{1}{2r-1}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}_T}{\mathcal{E}_t}\right)^{\frac{1}{r-1}}\right]\right)^{\frac{r-1}{2r-1}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leqslant s\leqslant T}(M_s)^{-\frac{2}{r-1}}\right]\mathbb{E}_t^{\mathbb{P}'}\left[(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'})^4\right]\right)^{\frac{r-1}{2(2r-1)}}\\
&\leqslant C\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}\right)^4\right]\right)^{\frac{r-1}{2(2r-1)}}(\delta Y_t)^{\frac{1}{2r-1}}.
\end{aligned}$$

By following the arguments of the proof of Theorem 2.3.1 (ii) and (iii), we then deduce the minimum condition. Finally, the fact that the processes $K^{\mathbb{P}}$ can be aggegated is a direct consequence of the general aggregation result of Theorem 5.1 in [103]. $\square$

Remark 2.4.1. *We emphasize that the result of Lemma* 2.1 *in [14] can only be applied when the generator is deterministic. However, even though F is indeed deterministic, $\widehat{F}$ is not, because $\widehat{a}$ is random. Nonetheless, given the particular form for the density of the quadratic variation of the canonical process we assumed in the definition of $\widetilde{\mathcal{P}}_H$, we can apply the result of Briand and Elie between the stopping times and on each set of the partition of Ω, since then $\widehat{a}$ and thus $\widehat{F}$ is indeed deterministic.*

2.5 A pathwise proof of existence

We have seen in the previous Section that it is usually extremely difficult to prove existence of a solution to a 2BSDE using monotone approximation techniques. Nonetheless, we have shown in Theorem 2.3.1 that if a solution exists, it will necessarily verify the representation (2.2.7). This gives us a natural candidate for the solution as a supremum of solutions to standard BSDEs. However, since those BSDEs are all defined on the support of mutually singular probability measures, it seems difficult to define such a supremum, because of the problems raised by the negligible sets. In order to overcome this, Soner, Touzi and Zhang proposed in [101] a pathwise construction of the solution to a 2BSDE. Let us describe briefly their strategy.

The first step is to define pathwise the solution to a standard BSDE. For simplicity, let us consider first a BSDE with a generator equal to 0. Then, we know that the solution is given by the conditional expectation of the terminal condition. In order to define this solution pathwise, we can use the so-called regular conditional probability distribution (r.p.c.d. for short) of Stroock and Varadhan [104]. In the general case, the idea is similar and consists on defining BSDEs on a shifted canonical space.

Finally, we have to prove measurability and regularity of the candidate solution thus obtained, and the decomposition (2.2.5) is obtained through a non-linear Doob-Meyer decomposition. Our aim in this section is to extend this approach to the quadratic case.

2.5.1 Notations

For the convenience of the reader, we recall below some of the notations introduced in [101].

- For $0 \leqslant t \leqslant T$, denote by $\Omega^t := \left\{\omega \in C\left([t,T],\mathbb{R}^d\right),\ w(t)=0\right\}$ the shifted canonical space, B^t the shifted canonical process, $\mathbb{P}_0^t$ the shifted Wiener measure and $\mathbb{F}^t$ the filtration generated by B^t. We define the density process $\widehat{a}^t$ of the quadratic variation process $\langle B^t\rangle$.

- For $0 \leqslant s \leqslant t \leqslant T$ and $\omega \in \Omega^s$, define the shifted path $\omega^t \in \Omega^t$
$$\omega_r^t := \omega_r - \omega_t,\ \forall r \in [t,T].$$

- For $0 \leqslant s \leqslant t \leqslant T$ and $\omega \in \Omega^s$, $\widetilde{\omega} \in \Omega^t$ define the concatenation path $\omega \otimes_t \widetilde{\omega} \in \Omega^s$ by
$$(\omega \otimes_t \widetilde{\omega})(r) := \omega_r 1_{[s,t)}(r) + (\omega_t + \widetilde{\omega}_r)1_{[t,T]}(r),\ \forall r \in [s,T].$$

- For $0 \leqslant s \leqslant t \leqslant T$ and a $\mathcal{F}_T^s$-measurable random variable ξ on Ω^s, for each $\omega \in \Omega^s$, define the shifted $\mathcal{F}_T^t$-measurable random variable $\xi^{t,\omega}$ on Ω^t by
$$\xi^{t,\omega}(\widetilde{\omega}) := \xi(\omega \otimes_t \widetilde{\omega}),\ \forall \widetilde{\omega} \in \Omega^t.$$
Similarly, for an $\mathbb{F}^s$-progressively measurable process X on $[s,T]$ and $(t,\omega) \in [s,T] \times \Omega^s$, the shifted process $\{X_r^{t,\omega}, r \in [t,T]\}$ is $\mathbb{F}^t$-progressively measurable.

- For a $\mathbb{F}$-stopping time τ, the r.c.p.d. of $\mathbb{P}$ (noted $\mathbb{P}_\tau^\omega$) induces naturally a probability measure $\mathbb{P}^{\tau,\omega}$ (that we also call the r.c.p.d. of $\mathbb{P}$) on $\mathcal{F}_T^{\tau(\omega)}$ which in particular satisfies that for every bounded and $\mathcal{F}_T$-measurable random variable ξ
$$\mathbb{E}^{\mathbb{P}_\tau^\omega}\left[\xi\right] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}\left[\xi^{\tau,\omega}\right].$$

- We define similarly as in Section 2.2 the set $\bar{\mathcal{P}}_S^t$, by restricting to the shifted canonical space Ω^t, and its subset $\mathcal{P}_H^t$.

- Finally, we define our "shifted" generator
$$\widehat{F}_s^{t,\omega}(\widetilde{\omega},y,z) := F_s(\omega \otimes_t \widetilde{\omega},y,z,\widehat{a}_s^t(\widetilde{\omega})),\ \forall (s,\widetilde{\omega}) \in [t,T] \times \Omega^t.$$

Notice that thanks to Lemma 4.1 in [102], this generator coincides for $\mathbb{P}$-a.e. ω with the shifted generator as defined above, that is to say
$$F_s(\omega \otimes_t \widetilde{\omega},y,z,\widehat{a}_s(\omega \otimes_t \widetilde{\omega})).$$

The advantage of the chosen "shifted" generator is that it inherits the uniform continuity in ω under the $\mathbb{L}^\infty$ norm of F.

2.5.2 Existence when ξ is in $\mathrm{UC_b}(\Omega)$

As mentioned at the beginning of the Section, we will need to prove some measurability and regularity on our candidate solution. For this purpose, we need to assume more regularity on the terminal condition. When ξ is in $\mathrm{UC_b}(\Omega)$, by definition there exists a modulus of continuity function ρ for ξ and F in ω. Then, for any $0 \leqslant t \leqslant s \leqslant T$, $(y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$ and $\omega,\omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$\left|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})\right| \leqslant \rho\left(\|\omega-\omega'\|_t\right) \text{ and } \left|\widehat{F}_s^{t,\omega}(\tilde{\omega},y,z) - \widehat{F}_s^{t,\omega'}(\tilde{\omega},y,z)\right| \leqslant \rho\left(\|\omega-\omega'\|_t\right),$$

where $\|\omega\|_t := \sup_{0\leqslant s\leqslant t}|\omega_s|$, $0 \leqslant t \leqslant T$.

To prove existence, as in [101], we define the following value process V_t pathwise:

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}_H^t} \mathcal{Y}_t^{\mathbb{P},t,\omega}(T,\xi), \text{ for all } (t,\omega) \in [0,T]\times\Omega, \tag{2.5.1}$$

where, for any $(t_1,\omega) \in [0,T]\times\Omega$, $\mathbb{P} \in \mathcal{P}_H^{t_1}$, $t_2 \in [t_1,T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta \in \mathbb{L}^\infty(\mathbb{P})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,\eta) := y_{t_1}^{\mathbb{P},t_1,\omega}$, where $\left(y^{\mathbb{P},t_1,\omega}, z^{\mathbb{P},t_1,\omega}\right)$ is the solution of the following BSDE on the shifted space Ω^{t_1} under $\mathbb{P}$

$$y_s^{\mathbb{P},t_1,\omega} = \eta^{t_1,\omega} - \int_s^{t_2} \widehat{F}_r^{t_1,\omega}\left(y_r^{\mathbb{P},t_1,\omega}, z_r^{\mathbb{P},t_1,\omega}\right)dr - \int_s^{t_2} z_r^{\mathbb{P},t_1,\omega}dB_r^{t_1},\ s \in [t_1,t_2],\ \mathbb{P}-\text{a.s.} \tag{2.5.2}$$

We recall that since the Blumenthal zero-one law holds for all our probability measures, $\mathcal{Y}_t^{\mathbb{P},t,\omega}(1,\xi)$ is constant for any given (t,ω) and $\mathbb{P} \in \mathcal{P}_H^t$. Therefore, the process V is well defined. However, we still do not know anything about its measurability. The following Lemma answers this question and explains the uniform continuity Assumptions in ω we made.

Lemma 2.5.1. *Let Assumptions 2.2.1 hold true and let ξ be in $\mathrm{UC_b}(\Omega)$. Then*

$$|V_t(\omega)| \leqslant C\left(1+\|\xi\|_{\mathbb{L}_H^\infty}\right), \textit{ for all } (t,\omega) \in [0,T]\times\Omega.$$

Furthermore

$$|V_t(\omega) - V_t(\omega')| \leqslant C\rho\left(\|\omega-\omega'\|_t\right), \textit{ for all } (t,\omega,\omega') \in [0,T]\times\Omega^2.$$

In particular, V_t is $\mathcal{F}_t$-measurable for every $t \in [0,T]$.

Proof. (i) For each $(t,\omega) \in [0,T]\times\Omega$ and $\mathbb{P} \in \mathcal{P}_H^t$, note that

$$\begin{aligned} y_s^{\mathbb{P},t,\omega} = \xi^{t,\omega} &- \int_s^T \left[\widehat{F}_r^{t,\omega}(0) + \lambda_r y_r^{\mathbb{P},t,\omega} + \eta_r\left(\widehat{a}_r^t\right)^{1/2} z_r^{\mathbb{P},t,\omega} + \phi_r\left(\widehat{a}_r^t\right)^{1/2} z_r^{\mathbb{P},t,\omega}\right]dr \\ &- \int_s^T z_r^{\mathbb{P},t,\omega}dB_r^t,\ s \in [t,T],\ \mathbb{P}-a.s. \end{aligned}$$

where λ is bounded and η satisfies

$$|\eta_r| \leqslant \mu\left|\widehat{a}_t^{1/2} z_r^{\mathbb{P},t,\omega}\right|,\ \mathbb{P}-a.s.$$

Then proceeding exactly as in the second step of the proof of Theorem 2.3.1, we can define a bounded process M and a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that

$$\left|y_t^{\mathbb{P},t,\omega}\right| \leqslant \mathbb{E}_t^{\mathbb{Q}}\left[M_T\left|\xi^{t,\omega}\right|\right] \leqslant C\left(1+\|\xi\|_{\mathbb{L}_H^\infty}\right).$$

By arbitrariness of $\mathbb{P}$, we get $|V_t(\omega)| \leqslant C(1+\|\xi\|_{\mathbb{L}_H^\infty})$.

(ii) The proof is exactly the same as above, except that we need to use uniform continuity in ω of $\xi^{t,\omega}$ and $\widehat{F}^{t,\omega}$. In fact, if we define for $(t,\omega,\omega') \in [0,T]\times\Omega^2$

$$\delta y := y^{\mathbb{P},t,\omega} - y^{\mathbb{P},t,\omega'},\ \delta z := z^{\mathbb{P},t,\omega} - z^{\mathbb{P},t,\omega'},\ \delta\xi := \xi^{t,\omega} - \xi^{t,\omega'},\ \delta\widehat{F} := \widehat{F}^{t,\omega} - \widehat{F}^{t,\omega'},$$

then we get with the same notations

$$|\delta y_t| = \mathbb{E}^{\mathbb{Q}}\left[M_T\delta\xi + \int_t^T M_s\delta\widehat{F}_s ds\right] \leqslant C\rho(\|\omega-\omega'\|_t).$$

We get the result by arbitrariness of $\mathbb{P}$. □

Then, we show the same dynamic programming principle as Proposition 4.7 in [102]

Proposition 2.5.1. *Let $\xi \in \mathrm{UC_b}(\Omega)$. Under Assumption 2.2.1 or Assumption 2.2.2 with the addition that the $\mathbb{L}_H^\infty$-norms of ξ and $\widehat{F}^0$ are small enough, we have for all $0 \leqslant t_1 < t_2 \leqslant T$ and for all $\omega \in \Omega$*

$$V_{t_1}(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2, V_{t_2}^{t_1,\omega}).$$

The proof is almost the same as the proof in [102], but we give it for the convenience of the reader.

Proof. Without loss of generality, we can assume that $t_1 = 0$ and $t_2 = t$. Thus, we have to prove

$$V_0(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

Denote $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(T,\xi), \mathcal{Z}^{\mathbb{P}}(T,\xi))$

(i) For any $\mathbb{P} \in \mathcal{P}_H$, it follows from Lemma 4.3 in [102], that for $\mathbb{P} - a.e.\ \omega \in \Omega$, the r.c.p.d. $\mathbb{P}^{t,\omega} \in \mathcal{P}_H^t$. By Tevzadze [107], we know that when the norm of the terminal condition and the norm of the generator valued on $(0,0)$ are small, a quadratic BSDE whose generator satisfies Assumption (2.2.2) (v) can be constructed via Picard iteration. Thus, it means that at each step of the iteration, the solution can be formulated as a conditional expectation under $\mathbb{P}$. Then, for general case, Tevzadze showed that if the generator satisfies Assumption (2.2.1) (v), the solution of the quadratic BSDE can be written as a sum of quadratic BSDEs with small terminal conditions and generators which are small on $(0,0)$. By the properties of the r.p.c.d., this implies that

$$y_t^{\mathbb{P}}(\omega) = \mathcal{Y}_t^{\mathbb{P}^{t,\omega},t,\omega}(T,\xi),\ \text{for } \mathbb{P} - a.e.\ \omega \in \Omega.$$

By definition of V_t and the comparison principle for quadratic BSDEs, we deduce that $y_0^{\mathbb{P}} \leqslant \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$ and it follows from the arbitrariness of $\mathbb{P}$ that

$$V_0(\omega) \leqslant \sup_{\mathbb{P} \in \mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

(ii) For the other inequality, we proceed as in [102]. Let $\mathbb{P} \in \mathcal{P}_H$ and $\varepsilon > 0$. The idea is to use the definition of V as a supremum to obtain an ε-optimizer. However, since V depends obviously on ω, we have to find a way to control its dependence in ω by restricting it in a small ball. But, since the canonical space is separable, this is easy. Indeed, there exists a partition $(E_t^i)_{i \geqslant 1} \subset \mathcal{F}_t$ such that $\|\omega - \omega'\|_t \leqslant \varepsilon$ for any i and any $\omega, \omega' \in E_t^i$.

Now for each i, fix an $\widehat{\omega}_i \in E_t^i$ and let, as advocated above, $\mathbb{P}_t^i$ be an $\varepsilon-$optimizer of $V_t(\widehat{\omega}_i)$. If we define for each $n \geqslant 1$, $\mathbb{P}^n := \mathbb{P}^{n,\varepsilon}$ by

$$\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_t^i}\left[1_E^{t,\omega}\right] 1_{E_t^i}\right] + \mathbb{P}(E \cap \widehat{E}_t^n), \text{ where } \widehat{E}_t^n := \cup_{i>n} E_t^i,$$

then, by the proof of Proposition 4.7 in [102], we know that $\mathbb{P}^n \in \mathcal{P}_H$ and that

$$V_t \leqslant y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon),\ \mathbb{P}^n - a.s. \text{ on } \cup_{i=1}^n E_t^i.$$

Let now $(y^n, z^n) := (y^{n,\varepsilon}, z^{n,\varepsilon})$ be the solution of the following BSDE on $[0, t]$

$$y_s^n = \left[y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon)\right] 1_{\cup_{i=1}^n E_t^i} + V_t 1_{\widehat{E}_t^n} - \int_s^t \widehat{F}_r(y_r^n, z_r^n)dr - \int_s^t z_r^n dB_r,\ \mathbb{P}^n - a.s. \quad (2.5.3)$$

Note that since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, the equality (2.5.3) also holds $\mathbb{P} - a.s.$ By the comparison theorem, we know that $\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leqslant y_0^n$. Using the same arguments and notations as in the proof of Lemma 2.5.1, we obtain

$$\left|y_0^n - y_0^{\mathbb{P}^n}\right| \leqslant C\mathbb{E}^{\mathbb{Q}}\left[\varepsilon + \rho(\varepsilon) + \left|V_t - y_t^{\mathbb{P}^n}\right| 1_{\widehat{E}_t^n}\right].$$

Then, by Lemma 2.5.1, we have

$$\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leqslant y_0^n \leqslant V_0(\omega) + C\left(\varepsilon + \rho(\varepsilon) + \mathbb{E}^{\mathbb{Q}}\left[\Lambda 1_{\widehat{E}_t^n}\right]\right).$$

The result follows from letting n go to $+\infty$ and ε to 0. □

Remark 2.5.1. *We want to emphasize here that it is only because of this Proposition proving the dynamic programming equation that we had to consider Tevzadze [107] approach to quadratic BSDEs, instead of the more classical approach of Kobylanski [63]. Indeed, as pointed out in the proof, for technical reasons we want to be able to construct solutions of BSDEs via Picard iterations, to build upon the known properties of the r.c.p.d. Using the Assumptions 2.2.1 or 2.2.2 with the addition that the $\mathbb{L}_H^\infty$-norms of ξ and $\widehat{F}^0$ are small enough, this allows us to recover this property.*

Now that we solved the measurability issues for V_t, we need to study its regularity in time. However, it seems difficult to obtain a result directly, given the definition of V. This is the reason why we define now for all (t,ω), the $\mathbb{F}^+$-progressively measurable process

$$V_t^+ := \overline{\lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}} V_r.$$

This new value process will then be proved to be càdlàg. Notice that a priori V^+ is only $\mathbb{F}^+$-progressively measurable, and not $\mathbb{F}$-progressively measurable. This explains why in the definition of the spaces in Section 2.2.4, the processes are assumed to be $\mathbb{F}^+$-progressively measurable.

Lemma 2.5.2. *Under the conditions of the previous Proposition, we have*

$$V_t^+ = \lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t} V_r,\ \mathcal{P}_H - q.s.$$

and thus V^+ is càdlàg $\mathcal{P}_H - q.s.$

Proof. Actually, we can proceed exactly as in the proof of Lemma 4.8 in [102], since the theory of g-expectations of Peng has been extended by Ma and Yao in [76] to the quadratic case (see in particular their Corollary 5.6 for our purpose). □

Finally, proceeding exactly as in Steps 1 and 2 of the proof of Theorem 4.5 in [102], and in particular using the Doob-Meyer decomposition proved in [76] (Theorem 5.2), we can get the existence of a universal process Z and a family of nondecreasing processes $\left\{K^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_H\right\}$ such that

$$V_t^+ = V_0^+ + \int_0^t \widehat{F}_s(V_s^+, Z_s)ds + \int_0^t Z_s dB_s - K_t^{\mathbb{P}},\ \mathbb{P}-a.s.\ \forall \mathbb{P}\in\mathcal{P}_H.$$

For the sake of completeness, we provide the representation (2.3.2) for V and V^+, and that, as shown in Proposition 4.11 of [102], we actually have $V = V^+$, $\mathcal{P}_H - q.s.$, which shows that in the case of a terminal condition in $UC_b(\Omega)$, the solution of the 2BSDE is actually $\mathbb{F}$-progressively measurable. This will be important in Section 2.7.

Proposition 2.5.2. *Let $\xi\in UC_b(\Omega)$. Under Assumption 2.2.1 or Assumption 2.2.2 with the addition that the $\mathbb{L}_H^\infty$-norms of ξ and $\widehat{F}^0$ are small enough, we have*

$$V_t = \underset{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi) \text{ and } V_t^+ = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi),\ \mathbb{P}-a.s.,\ \forall\mathbb{P}\in\mathcal{P}_H.$$

Besides, we also have for all t, $V_t = V_t^+$, $\mathcal{P}_H - q.s.$

Proof. The proof for the representations is the same as the proof of proposition 4.10 in [102], since we also have a stability result for quadratic BSDEs under our assumptions. For the equality between V and V^+, we also refer to the proof of Proposition 4.11 in [102]. □

To be sure that we have found a solution to our 2BSDE, it remains to check that the family of nondecreasing processes above satisfies the minimum condition. Let $\mathbb{P}\in\mathcal{P}_H$,

$t \in [0,T]$ and $\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})$. From the proof of Theorem 2.3.1, we have with the same notations

$$\delta V_t = \mathbb{E}_t^{\mathbb{Q}'}\left[\int_t^T M_t dK_t^{\mathbb{P}'}\right] \geqslant \mathbb{E}_t^{\mathbb{Q}'}\left[\inf_{t\leqslant s\leqslant T}(M_s)(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'})\right]$$
$$= \frac{\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\inf_{t\leqslant s\leqslant T}(M_s)(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'})\right]}{\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\right]}$$

For notational convenience, denote $\mathcal{E}_t := \mathcal{E}\left(\int_0^t(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)$. Let r be the number given by Lemma 2.2.2 applied to $\mathcal{E}$. Then we estimate

$$\mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right]$$
$$\leqslant \mathbb{E}_t^{\mathbb{P}'}\left[\frac{\mathcal{E}_T}{\mathcal{E}_t}\inf_{t\leqslant s\leqslant T}(M_s)(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'})\right]^{\frac{1}{2r-1}}\mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}_t}{\inf_{t\leqslant s\leqslant T}(M_s)\mathcal{E}_T}\right)^{\frac{1}{2(r-1)}}(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'})\right]^{\frac{2(r-1)}{2r-1}}$$
$$\leqslant (\delta V_t)^{\frac{1}{2r-1}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}_t}{\mathcal{E}_T}\right)^{\frac{1}{r-1}}\right]\right)^{\frac{r-1}{2r-1}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leqslant s\leqslant T}(M_s)^{-\frac{2}{r-1}}\right]\mathbb{E}_t^{\mathbb{P}'}\left[(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'})^4\right]\right)^{\frac{r-1}{2(2r-1)}}$$
$$\leqslant C\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}\right)^4\right]\right)^{\frac{r-1}{2(2r-1)}}(\delta V_t)^{\frac{1}{2r-1}}.$$

By following the arguments of the proof of Theorem 2.3.1 (ii) and (iii), we then deduce the minimum condition.

Remark 2.5.2. *In order to prove the minimum condition it is fundamental that the process M above is bounded from below. For instance, it would not be the case if we had replaced the Lipschitz assumption on y by a monotonicity condition as in [90].*

2.5.3 Main result

We are now in position to state the main result of this section

Theorem 2.5.1. *Let $\xi \in \mathcal{L}_H^\infty$. Under Assumption 2.2.1 or Assumption 2.2.2 with the addition that the $\mathbb{L}_H^\infty$-norms of ξ and $\widehat{F}^0$ are small enough, there exists a unique solution $(Y,Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the* 2BSDE (2.2.5).

Proof. For $\xi \in \mathcal{L}_H^\infty$, there exists $\xi_n \in \mathrm{UC_b}(\Omega)$ such that $\|\xi-\xi_n\| \underset{n\to+\infty}{\longrightarrow} 0$. Then, thanks to the *a priori* estimates obtained in Proposition 2.3.2, we can proceed exactly as in the proof of Theorem 4.6 (ii) in [101] to obtain the solution as a limit of the solution of the 2BSDE (2.2.5) with terminal condition ξ_n. □

2.6 An application to robust risk-sensitive control

One application of classical quadratic BSDEs is to study risk-sensitive control problems, see El Karoui, Hamadène et Matoussi [35] for more details. In this section, we will consider a robust version of these problems.

First of all, for technical reasons, we restrict the probability measures in $\widetilde{\mathcal{P}}_H := \widetilde{\mathcal{P}}_S \bigcap \mathcal{P}_H$, where $\widetilde{\mathcal{P}}_S$ is defined in Subsection 2.2.1. Then $\widehat{a}$ is uniformly bounded by some $\overline{a}$, $\underline{a} \in \mathbb{S}_d^{>0}$.

For each $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, we can define a $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$ by

$$dW_t^{\mathbb{P}} = \widehat{a}_t^{-1/2} dB_t \ \mathbb{P} - a.s.$$

Let us now consider some system, whose evolution is decribed (for simplicity) by the canonical process B. A controller then intervenes on the system via an adapted stochastic process u which takes its values in a compact metric space U. The set of those controls is called admissible and denoted by $\mathcal{U}$. When the controller acts with u under the probability $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, the dynamic of the controlled system remains the same, but now under the probability measure $\mathbb{P}^u$ defined by its density with respect to $\mathbb{P}$

$$\frac{d\mathbb{P}^u}{d\mathbb{P}} = \exp\left(\int_0^T \widehat{a}_t^{-1/2} g(t, B_\cdot, u_t) dW_t^{\mathbb{P}} - \frac{1}{2}\int_0^T \left|\widehat{a}_t^{-1/2} g(t, B_\cdot, u_t)\right|^2 dt\right),$$

where $g(t, \omega, u)$ is assumed to be bounded, continuous with respect to u, adapted and uniformly continuous in ω. Notice that this probability measure is well defined since $\widehat{a}$ is uniformly bounded.

Then, under $\mathbb{P}^u$, the dynamic of the system is given by

$$dB_t = g(t, B_\cdot, u_t)dt + \widehat{a}_t^{1/2} dW^{\mathbb{P},u}, \ \mathbb{P}^u - a.s.$$

where $W^{\mathbb{P},u}$ is a Brownian motion under $\mathbb{P}^u$ defined by

$$dW_t^{\mathbb{P},u} = dW_t^{\mathbb{P}} - \widehat{a}_t^{-1/2} g(t, B_\cdot, u_t)dt.$$

When the controller is risk seeking, we assume that the reward functional of the control action is given by the following expression

$$\forall u \in \mathcal{U}, \ J(u) := \sup_{\mathbb{P} \in \widetilde{\mathcal{P}}_H} \mathbb{E}^{\mathbb{P},u}\left[\exp\left(\theta \int_0^T h(s, B_\cdot, u_s)ds + \Psi(B_T)\right)\right]$$

where $\theta > 0$ is a real parameter which represents the sensitiveness of the controller with respect to risk. Here $h(t, \omega, u)$ is assumed to adapted and continuous in u, and both Ψ and h are assumed to be bounded and uniformly continuous in ω for the $\|\cdot\|_\infty$ norm. We are interested in finding an admissible control u^* which maximizes the reward $J(u)$ for the controller.

We begin with establishing the link between $J(u)$ and 2BSDEs in the following proposition

Proposition 2.6.1. *There exists a unique solution (Y^u, Z^u) of the 2BSDE associated with the generator $-zg(t, B_\cdot, u_t) - h(t, B_\cdot, u_t) - \frac{\theta}{2}|\widehat{a}_t^{1/2} z|^2$, i.e., $\mathbb{P}-a.s.$, for all $\mathbb{P} \in \widetilde{\mathcal{P}}_H$*

$$Y_t^u = \Psi(B_T) + \int_t^T \left(Z_s^u g(s, B_\cdot, u_s) + h(s, B_\cdot, u_s) + \frac{\theta}{2}|\widehat{a}_s^{1/2} Z_s^u|^2 \right) ds - \int_t^T Z_s^u dB_s - dK_t^{u,\mathbb{P}}. \tag{2.6.1}$$

Moreover $J(u) = \exp(\theta Y_0^u)$.

Proof. With our assumptions on g, h and Ψ, we know that the generator satisfies the Assumption 2.2.1, therefore there exists a unique solution to the 2BSDE (2.6.1). According to [35], the solution to the classical BSDE with the same terminal condition and generator as the 2BSDE (2.6.1) under each $\mathbb{P}$ is

$$y_t^{u,\mathbb{P}} = \frac{1}{\theta} \ln \left(\mathbb{E}_t^{\mathbb{P},u} \left[\exp \left(\theta \int_t^T h(s, B_\cdot, u_s) ds + \Psi(B_T) \right) \right] \right), \ \mathbb{P}-a.s.$$

Then by the representation for Y^u, we have

$$Y_t^u = \frac{1}{\theta} \operatorname*{ess\,sup}_{\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+, \mathbb{P})}^{\mathbb{P}} \ln \left(\mathbb{E}_t^{\mathbb{P},u} \left[\exp \left(\theta \int_t^T h(s, B_\cdot, u_s) ds + \Psi(B_T) \right) \right] \right), \mathbb{P}-a.s.$$

Since the functional $\ln(x)$ is monotone non-decreasing, then

$$Y_t^u = \frac{1}{\theta} \ln \left(\operatorname*{ess\,sup}_{\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'u} \left[\exp \left(\theta \int_t^T h(s, B_\cdot, u_s) ds + \Psi(B_T) \right) \right] \right), \ \mathbb{P}-a.s.$$

Therefore, we have $J(u) = \exp\{\theta Y_0^u\}$. □

As explained in [35], by applying Benes' selection theorem, there exists a measurable version $u^*(t, B_\cdot, z)$ of

$$\arg\max \ I(t, B_\cdot, z, u) := zg(t, B_\cdot, u) + h(t, B_\cdot, u).$$

We know that $I^*(t, B_\cdot, z) := \sup_{u \in U} I(t, B_\cdot, z, u) = I(t, B_\cdot, z, u^*(t, B_\cdot, z))$ is convex uniformly Lipschitz in z because it is the supremum of functions which are linear in z. So the mapping $z \to I^*(t, B_\cdot, z) + \frac{1}{2}|\widehat{a}_t^{1/2} z|^2$ is continuous with quadratic growth, implying that a solution $(y^{*,\mathbb{P}}, z^{*,\mathbb{P}})$ of the BSDE associated to this generator exists. Then we have

Theorem 2.6.1. *There exists a unique solution (Y^*, Z^*) to the following 2BSDE*

$$Y_t^* = \Psi(B_T) + \int_t^T \left(I^*(s, B_\cdot, Z_s^*) + \frac{\theta}{2}|\widehat{a}_s^{1/2} Z_s^*|^2 \right) ds - \int_t^T Z_s^* dB_s + K_T^{*,\mathbb{P}} - K_t^{*,\mathbb{P}}. \tag{2.6.2}$$

The admissible control $u^* := (u^*(t, B_\cdot, Z_t^*))_{t \leqslant T}$ is optimal and $(\exp(Y_t^*))_{t \leqslant T}$ is the value function of the robust risk-sensitive control problem, i.e., for any $t \leqslant T$ we have:

$$\exp(Y_t^*) = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}^{\mathbb{P}} \operatorname*{ess\,sup}_{u \in \mathcal{U}}^{\mathbb{P}} \mathbb{E}_t^{\mathbb{P}',u} \left[\exp \left(\theta \int_t^T h(s, B_\cdot, u_s) ds + \Psi(B_T) \right) \right].$$

Proof. First, we need to prove the existence of a solution to the quadratic 2BSDE (2.6.2). Unlike in Proposition 2.6.1, here u^* also depends on z, so we do not know whether I^* is twice differentiable with respect to z. Therefore the generator of the 2BSDE may not satisfy the Assumption 2.2.1. But it's easy to see that it always satisfies the weaker Assumption 2.2.2, and we only need this Assumption to have uniqueness of the solution. Moreover, it was also the only one used to prove the minimum condition for the familly of non-decreasing processes in Subsection 2.5.2. Therefore, exactly as in Section 2.4, for $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, by making the exponential change

$$\overline{Y}_t := e^{\theta Y_t^*},\ \overline{Z}_t := \theta \overline{Y}_t Z_t^*,\ \overline{K}_t^{\mathbb{P}} := \theta \int_0^t \overline{Y}_s dK_s^{*,\mathbb{P}} - \sum_{0 \leqslant s \leqslant t} e^{\theta Y_s^*} - e^{\theta Y_{s^-}^*} - \theta \Delta Y_s^* e^{\theta Y_{s^-}^*},$$

we see that $(\overline{Y}, \overline{Z}, \overline{K}^{\mathbb{P}})$ formally verifies the following equation

$$\overline{Y}_t = e^{\theta \Psi(B_T)} + \int_t^T \sup_{u \in U} \left\{ \overline{Z}_s g(s, B_\cdot, u) + \theta \overline{Y}_s h(s, B_\cdot, u) \right\} ds - \int_t^T \overline{Z}_s dB_s + \overline{K}_T^{\mathbb{P}} - \overline{K}_t^{\mathbb{P}},\ \mathbb{P}-a.s. \tag{2.6.3}$$

Since this is 2BSDE with Lipschitz generator from Soner, Touzi and Zhang [101], we know that $(\overline{Y}, \overline{Z}, \overline{K}^{\mathbb{P}})$ exists, is unique and satisfies the representation property (2.3.2). Arguing exactly as in Subsection 2.4.1 for the purely quadratic 2BSDEs, we can then obtain the existence. Now, from [35], we have that

$$\exp\left(y_t^{*,\mathbb{P}}\right) = \operatorname*{ess\,sup}^{\mathbb{P}}_{u \in \mathcal{U}} \mathbb{E}_t^{\mathbb{P}^u} \left[\exp\left(\theta \int_t^T h(s, B_\cdot, u_s) ds + \Psi(B_T) \right) \right].$$

Then the representation for Y^* implies the desired result. □

2.7 Connection with fully nonlinear PDEs

In this section, we place ourselves in the general case of Section 2.2, and we assume moreover that all the nonlinearity in H only depends on the current value of the canonical process B (the so-called Markov property)

$$H_t(\omega, y, z, \gamma) = h(t, B_t(\omega), y, z, \gamma),$$

where $h : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times D_h \to \mathbb{R}$ is a deterministic map. Then, we define as in Section 2.2 the corresponding conjugate and bi-conjugate functions

$$f(t, x, y, z, a) := \sup_{\gamma \in D_h} \left\{ \frac{1}{2} \mathrm{Tr}\,[a\gamma] - h(t, x, y, z, \gamma) \right\} \tag{2.7.1}$$

$$\widehat{h}(t, x, y, z, \gamma) := \sup_{a \in \mathbb{S}_d^{>0}} \left\{ \frac{1}{2} \mathrm{Tr}\,[a\gamma] - f(t, x, y, z, a) \right\} \tag{2.7.2}$$

We denote $\mathcal{P}_h := \mathcal{P}_H$, and following [101], we strengthen Assumption 2.2.1

Assumption 2.7.1. (i) *$\mathcal{P}_h$ is not empty, and the domain D_{f_t} of the map $a \to f(t, x, y, z, a)$ is independent of (x, y, z).*

(ii) *On D_{f_t}, f is uniformly continuous in t, uniformly in a.*

(iii) *f is continuous in z and has the following growth property. There exists (α, β, γ) such that*

$$|f(t,x,y,z,a)| \leqslant \alpha + \beta |y| + \frac{\gamma}{2} \left|a^{1/2}z\right|^2, \text{ for all } t \in [0,T], x, z \in \mathbb{R}^d, y \in \mathbb{R}, a \in D_{f_t}.$$

(iv) *f is C^1 in y and C^2 in z, and there are constants r and θ such that for all $t \in [0,T], x, z \in \mathbb{R}^d, y \in \mathbb{R}, a \in D_{f_t}$*

$$|D_y f(t,x,y,z,a)| \leqslant r, \ |D_z f(t,x,y,z,a)| \leqslant r + \theta \left|a^{1/2}z\right|$$

$$|D^2_{zz} f(t,x,y,z,a)| \leqslant \theta.$$

(v) *On D_{f_t}, f is uniformly continuous in x, uniformly in (t,y,z,a), with a modulus of continuity ρ which has polynomial growth.*

Remark 2.7.1. *As mentioned in Subsection 2.2.3, when the norm of the terminal condition and the norm of $f(\cdot,0,0,a)$ are small enough, Assumption 2.7.1* (iv) *can be replaced by the following weaker assumptions.*

(iv')[a] *There exists $\mu > 0$ and a bounded $\mathbb{R}^d$-valued function ϕ such that for all $t \in [0,T], x, z, z' \in \mathbb{R}^d, y \in \mathbb{R}, a \in D_{f_t}$*

$$\left|f(t,x,y,z,a) - f(t,x,y,z',a) - \phi(t).a^{1/2}(z - z')\right| \leqslant \mu a^{1/2} \left|z - z'\right| \left(\left|a^{1/2}z\right| + \left|a^{1/2}z'\right|\right).$$

(iv')[b] *On D_{f_t}, f is Lipschitz in y, uniformly in (t,x,z,a).*

Let now $g : \mathbb{R}^d \to \mathbb{R}$ be a Lebesgue measurable and bounded function. Our object of interest here is the following Markovian 2BSDE with terminal condition $\xi = g(B_T)$

$$Y_t = g(B_T) - \int_t^T f(s, B_s, Y_s, Z_s, \widehat{a}_s) ds - \int_t^T Z_s dB_s + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \ \mathcal{P}_h - q.s. \qquad (2.7.3)$$

The aim of this section is to generalize the results of [101] and establish the connection $Y_t = v(t, B_t)$, $\mathcal{P}_h - q.s.$, where v is the solution in some sense of the following fully nonlinear PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) + \widehat{h}\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) = 0, \ t \in [0,T) \\ v(T,x) = g(x). \end{cases} \qquad (2.7.4)$$

Following the classical terminology in the BSDE literature, we say that the solution of the 2BSDE is Markovian if it can be represented by a deterministic function of t and B_t. In this subsection, we will construct such a function following the same spirit as in the construction in the previous section.

With the same notations for shifted spaces, we define for any $(t,x) \in [0,T] \times \mathbb{R}^d$

$$B_s^{t,x} := x + B_s^t, \text{ for all } s \in [t,T].$$

Let now τ be an $\mathbb{F}^t$-stopping time, $\mathbb{P} \in \mathcal{P}_h^t$ and η a $\mathbb{P}$-bounded $\mathcal{F}_\tau^t$-measurable random variable. Similarly as in (2.5.2), we denote $(y^{\mathbb{P},t,x}, z^{\mathbb{P},t,x}) := (\mathcal{Y}^{\mathbb{P},t,x}(\tau,\eta), \mathcal{Z}^{\mathbb{P},t,x}(\tau,\eta))$ the unique solution of the following BSDE

$$y_s^{\mathbb{P},t,x} = \eta - \int_s^\tau f(u, B_u^{t,x}, y_u^{\mathbb{P},t,x}, z_u^{\mathbb{P},t,x}, \widehat{a}_u^t)du - \int_s^\tau z_u^{\mathbb{P},t,x} dB_u^{t,x},\ t \leqslant s \leqslant \tau,\ \mathbb{P}-a.s. \quad (2.7.5)$$

Next, we define the following deterministic function (by virtue of the Blumenthal $0-1$ law)

$$u(t,x) := \sup_{\mathbb{P} \in \mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(T, g(B_T^{t,x})), \text{ for } (t,x) \in [0,T] \times \mathbb{R}^d. \quad (2.7.6)$$

We then have the following Theorem, which is actually Theorem 5.9 of [101] in our framework

Theorem 2.7.1. *Let Assumption 2.7.1 hold, and assume that g is bounded and uniformly continuous. Then the 2BSDE (2.7.3) has a unique solution $(Y,Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ and we have $Y_t = u(t, B_t)$. Moreover, u is uniformly continuous in x, uniformly in t and right-continuous in t.*

Proof. The existence and uniqueness for the 2BSDE follows directly from Theorem 2.5.1. Since $\xi \in \mathrm{UC_b}(\Omega)$, we have with the notations of the previous section $V_t = u(t, B_t)$. But, by Proposition 2.5.2, we know that $Y_t = V_t$, hence the first result.

Then the uniform continuity of u is a simple consequence of Lemma 2.5.1. Finally, the right-continuity of u in t can be obtained exactly as in the proof of Theorem 5.9 in [101]. $\Box$

2.7.1 Nonlinear Feynman-Kac formula in the quadratic case

Exactly as in the classical case and as in Theorem 5.3 in [101], we have a nonlinear version of the Feynman-Kac formula. The proof is the same as in [101], so we omit it. Notice however that it is more involved than in the classical case, mainly due to the technicalities introduced by the quasi-sure framework.

Theorem 2.7.2. *Let Assumption 2.7.1 hold true. Assume further that $\widehat{h}$ is continuous in its domain, that D_f is independent of t and is bounded both from above and away from 0. Let $v \in C^{1,2}([0,T), \mathbb{R}^d)$ be a classical solution of (2.7.4) with $\{(v, Dv)(t, B_t)\}_{0 \leqslant t \leqslant T} \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$. Then*

$$Y_t := v(t, B_t),\ Z_t := Dv(t, B_t),\ K_t := \int_0^t k_s ds,$$

is the unique solution of the quadratic 2BSDE (2.7.3), where

$$k_t := \widehat{h}(t, B_t, Y_t, Z_t, \Gamma_t) - \frac{1}{2}\mathrm{Tr}\left[\widehat{a}_t^{1/2}\Gamma_t\right] + f(t, B_t, Y_t, Z_t, \widehat{a}_t) \text{ and } \Gamma_t := D^2 v(t, B_t).$$

2.7.2 The viscosity solution property

As usual when dealing with possibly discontinuous viscosity solutions, we introduce the following upper and lower-semicontinuous envelopes

$$
\begin{aligned}
u_*(t,x) &:= \underline{\lim}_{(t',x')\to(t,x)} u(t',x'),\ u^*(t,x) := \overline{\lim}_{(t',x')\to(t,x)} u(t',x') \\
\widehat{h}_*(\vartheta) &:= \underline{\lim}_{(\vartheta')\to(\vartheta)} \widehat{h}(\vartheta'),\ \widehat{h}^*(\vartheta) := \overline{\lim}_{(\vartheta')\to(\vartheta)} \widehat{h}(\vartheta')
\end{aligned}
$$

In order to prove the main Theorem of this subsection, we will need the following Proposition, whose proof (which is rather technical) is omitted, since it is exactly the same as the proof of Propositions 5.10 and 5.14 and Lemma 6.2 in [101].

Proposition 2.7.1. *Let Assumption 2.7.1 hold. Then for any bounded function g*

(i) *For any (t,x) and arbitrary $\mathbb{F}^t$-stopping times $\left\{\tau^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_h^t\right\}$, we have*

$$u(t,x) \leqslant \sup_{\mathbb{P}\in\mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(\tau^{\mathbb{P}}, u^*(\tau^{\mathbb{P}}, B_{\tau^{\mathbb{P}}}^{t,x})).$$

(ii) *If in addition g is lower-semicontinuous, then*

$$u(t,x) = \sup_{\mathbb{P}\in\mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(\tau^{\mathbb{P}}, u(\tau^{\mathbb{P}}, B_{\tau^{\mathbb{P}}}^{t,x})).$$

Now we can state the main Theorem of this section

Theorem 2.7.3. *Let Assumption 2.7.1 hold true. Then*

(i) *u is a viscosity subsolution of*

$$-\partial_t u^* - \widehat{h}^*(\cdot, u^*, Du^*, D^2u^*) \leqslant 0,\ \text{on } [0,T)\times\mathbb{R}^d.$$

(ii) *If in addition g is lower-semicontinuous and D_f is independent of t, then u is a viscosity supersolution of*

$$-\partial_t u_* - \widehat{h}_*(\cdot, u_*, Du_*, D^2u_*) \geqslant 0,\ \text{on } [0,T)\times\mathbb{R}^d.$$

Proof. The proof follows closely the proof of Theorem 5.11 in [101], with some minor modifications (notably when we prove (2.7.10)). We provide it for the convenience of the reader.

(i) Assume to the contrary that

$$0 = (u^*-\phi)(t_0,x_0) > (u^*-\phi)(t,x) \text{ for all } (t,x)\in[0,T)\times\mathbb{R}^d\setminus\{(t_0,x_0)\}, \tag{2.7.7}$$

for some $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ and

$$\left(-\partial_t \phi - \widehat{h}^*(\cdot, \phi, D\phi, D^2\phi)\right)(t_0, x_0) > 0, \tag{2.7.8}$$

for some smooth and bounded function ϕ (we can assume w.l.o.g. that ϕ is bounded since we are working with bounded solutions of 2BSDEs).

Now since ϕ is smooth and since by definition $\widehat{h}^*$ is upper-semicontinuous, there exists an open ball $\mathcal{O}(r, (t_0, x_0))$ centered at (t_0, x_0) with radius r, which can be chosen less than $T - t_0$, such that

$$-\partial_t \phi - \widehat{h}(\cdot, \phi, D\phi, D^2\phi) \geqslant 0, \text{ on } \mathcal{O}(r, (t_0, x_0)).$$

By definition of $\widehat{h}$, this implies that for any $\alpha \in \mathbb{S}_d^{>0}$

$$-\partial_t \phi - \frac{1}{2}\mathrm{Tr}\left[\alpha D^2\phi\right] + f(\cdot, \phi, D\phi, \alpha) \geqslant 0, \text{ on } \mathcal{O}(r, (t_0, x_0)). \tag{2.7.9}$$

Let us now denote

$$\mu := -\max_{\partial\mathcal{O}(r,(t_0,x_0))} (u^* - \phi).$$

By (2.7.7), this quantity is strictly positive.

Let now (t_n, x_n) be a sequence in $\mathcal{O}(r, (t_0, x_0))$ such that $(t_n, x_n) \rightarrow (t_0, x_0)$ and $u(t_n, x_n) \rightarrow u^*(t_0, x_0)$. Denote the following stopping time

$$\tau_n := \inf\left\{s > t_n,\ (s, B_s^{t_n,x_n} \notin \mathcal{O}(r, (t_0, x_0))\right\}.$$

Since $r < T - t_0$, we have $\tau_n < T$ and therefore $(\tau_n, B_{\tau_n}^{t_n,x_n}) \in \partial\mathcal{O}(r, (t_0, x_0))$. Hence, we have

$$c_n := (\phi - u)(t_n, x_n) \rightarrow 0 \text{ and } u^*(\tau_n, B_{\tau_n}^{t_n,x_n}) \leqslant \phi(\tau_n, B_{\tau_n}^{t_n,x_n}) - \mu.$$

Fix now some $\mathbb{P}^n \in \mathcal{P}_h^{t_n}$. By the comparison Theorem for quadratic BSDEs, we have

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, u^*(\tau_n, B_{\tau_n}^{t_n,x_n})) \leqslant \mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n}) - \mu).$$

Then proceeding exactly as in the second step of the proof of Theorem 2.3.1, we can define a bounded process M_n, whose bounds only depend on T and the Lipschitz constant of f in y, and a probability measure $\mathbb{Q}_n$ equivalent to $\mathbb{P}_n$ such that

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n}) - \mu) - \mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n})) = -\mathbb{E}_{t_n}^{\mathbb{Q}_n}[M_{\tau_n}\mu] \leqslant -\mu',$$

for some strictly positive constant μ' which is independent of n.

Hence, we obtain by definition of c_n

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, u^*(\tau_n, B_{\tau_n}^{t_n,x_n})) - u(t_n, x_n) \leqslant \mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n})) - \phi(t_n, x_n) + c_n - \mu'. \tag{2.7.10}$$

With the same arguments as above, it is then easy to show with Itô's formula that

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n,\phi(\tau_n,B_{\tau_n}^{t_n,x_n}))-\phi(t_n,x_n)=\mathbb{E}_{t_n}^{\mathbb{Q}_n}\left[-\int_{t_n}^{\tau_n}M_s^n\psi_s^n ds\right],$$

where

$$\psi_s^n:=(-\partial_t\phi-\frac{1}{2}\mathrm{Tr}\left[\widehat{a}_s^t D^2\phi\right]+f(\cdot,D\phi,\widehat{a}_s^t))(s,B_s^{t_n,x_n}).$$

But by (2.7.9) and the definition of τ_n, we know that for $t_n\leqslant s\leqslant\tau_n$, $\psi_s^n\geqslant 0$. Recalling (2.7.10), we then get

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n,u^*(\tau_n,B_{\tau_n}^{t_n,x_n}))-u(t_n,x_n)\leqslant c_n-\mu'.$$

Since c_n does not depend on $\mathbb{P}_n$, we immediately get

$$\sup_{\mathbb{P}\in\mathcal{P}_h^{t_n}}\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n,u^*(\tau_n,B_{\tau_n}^{t_n,x_n}))-u(t_n,x_n)\leqslant c_n-\mu'.$$

The right-hand side is strictly negative for n large enough, which contradicts Proposition 2.7.1(i).

(ii) We also proceed by contradiction. Assuming to the contrary that

$$0=(u_*-\phi)(t_0,x_0)<(u_*-\phi)(t,x)\text{ for all }(t,x)\in[0,T)\times\mathbb{R}^d\setminus\{(t_0,x_0)\},\qquad(2.7.11)$$

for some $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$ and

$$\left(-\partial_t\phi-\widehat{h}_*(\cdot,\phi,D\phi,D^2\phi)\right)(t_0,x_0)<0,\qquad(2.7.12)$$

for some smooth and bounded function ϕ (we can assume w.l.o.g. that ϕ is bounded since we are working with bounded solutions of 2BSDEs).

Now we have by definition $\widehat{h}_*\leqslant\widehat{h}$, hence

$$\left(-\partial_t\phi-\widehat{h}(\cdot,\phi,D\phi,D^2\phi)\right)(t_0,x_0)<0,\qquad(2.7.13)$$

Unlike with the subsolution property, we do not know whether $D^2\phi(t_0,x_0)\in D_{\widehat{h}}$ or not. If it is the case, then by the definition of $\widehat{h}$, there exists some $\bar{\alpha}\in\mathbb{S}_d^{>0}$ such that

$$\left(-\partial_t\phi-\frac{1}{2}\mathrm{Tr}\left[\bar{\alpha}D^2\phi\right]+f(\cdot,\phi,D\phi,\bar{\alpha})\right)(t_0,x_0)<0,\qquad(2.7.14)$$

which implies in particular that $\bar{\alpha}\in D_f$.

If $D^2\phi(t_0,x_0)\notin D_{\widehat{h}}$, we still have that $\partial_t\phi(t_0,x_0)$ is finite, and thus $\bar{\alpha}\in D_f$ and (2.7.13) holds.

Now since ϕ is smooth and since D_f does not depend on t, there exists an open ball $\mathcal{O}(r,(t_0,x_0))$ centered at (t_0,x_0) with radius r, which can be chosen less than $T-t_0$, such that

$$-\partial_t\phi-\frac{1}{2}\mathrm{Tr}\left[\bar{\alpha}D^2\phi\right]+f(\cdot,\phi,D\phi,\bar{\alpha})\leqslant 0,\ \text{ on }\mathcal{O}(r,(t_0,x_0)).$$

Let us now denote

$$\mu := \min_{\partial\mathcal{O}(r,(t_0,x_0))} (u_* - \phi).$$

By (2.7.11), this quantity is strictly positive.

Let now (t_n, x_n) be a sequence in $\mathcal{O}(r,(t_0,x_0))$ such that $(t_n, x_n) \to (t_0, x_0)$ and $u(t_n, x_n) \to u_*(t_0, x_0)$. Denote the following stopping time

$$\tau_n := \inf\left\{s > t_n,\ (s, B_s^{t_n,x_n} \notin \mathcal{O}(r,(t_0,x_0))\right\}.$$

Since $r < T - t_0$, we have $\tau_n < T$ and therefore $(\tau_n, B_{\tau_n}^{t_n,x_n}) \in \partial\mathcal{O}(r,(t_0,x_0))$. Hence, we have

$$c_n := (\phi - u)(t_n, x_n) \to 0 \text{ and } u_*(\tau_n, B_{\tau_n}^{t_n,x_n}) \geqslant \phi(\tau_n, B_{\tau_n}^{t_n,x_n}) + \mu.$$

Now for each n consider the probability measure $\bar{\mathbb{P}}^n := \mathbb{P}^{\bar{\alpha}}$ induced by the constant diffusion $\bar{\alpha}$ from time t_n onwards. It is clearly in $\mathcal{P}_h^{t_n}$. Then, arguing exactly as in (i), we prove that

$$u(t_n, x_n) - \mathcal{Y}_{t_n}^{\bar{\mathbb{P}}^n,t_n,x_n}(\tau_n, u_*(\tau_n, B_{\tau_n}^{t_n,x_n})) \leqslant c_n - \mu',\ \bar{\mathbb{P}}^n - a.s.$$

For n large enough, the right-hand side becomes strictly negative, which contradicts Proposition 2.7.1(ii). □

Chapitre 3

Robust Utility Maximization in Non-dominated Models with 2BSDEs

3.1 Introduction

In this chapter, we study the problem of robust utility maximization with closed constraints set in uncertain volatility models via quadratic 2BSDEs introduced in Chapter 2. The rest of the chapter is organized as follows. In Section 3.2, we recall some notations of quadratic 2BSDEs. Then inspired by [38] and [54], in Sections 3.3, 3.4, 3.5 and 3.6, we study the problem for robust exponential utility, robust power utility and robust logarithmic utility. Finally, in Section 3.7, we provide some examples where we can explicitly solve the robust utility maximization problems by finding the solution of the associated 2BSDEs, and we give some insights and comparisons with the classical dynamic programming approach adopted in the seminal work of Merton [81]. This chapter is based on [78].

3.2 Preliminaries

We will use the notations and notions related to the theory of 2BSDEs with quadratic growth generators. The only difference is with the non-dominated family of mutually singular probability measures. We fix $\underline{a}, \overline{a} \in \mathbb{S}_d^{>0}$ such that $\underline{a} \leqslant \overline{a}$ (for the usual order on positive definite matrices, i.e. $(\overline{a} - \underline{a}) \in \mathbb{S}_d^{>0}$) and we define the family:

$$\mathcal{P}_H = \mathcal{P} := \left\{ \mathbb{P} \in \overline{\mathcal{P}}_S \text{ s.t. } \underline{a} \leqslant \widehat{a} \leqslant \overline{a}, dt \times d\mathbb{P} - a.e. \right\}.$$

In fact, this reduces to a particular case of Definition 2.2.1 in Chapter 2 where the bounds on $\widehat{a}$ are independent of the probability measures and where $\widehat{F}^0$ is bounded. Throughout this chapter we assume that $\mathcal{P}_H$ is not empty.

Definition 3.2.1. *We say a property holds $\mathcal{P}_H$-quasi-surely ($\mathcal{P}_H$-q.s. for short) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$.*

Remark 3.2.1. *The filtration $\mathbb{F}^+$ defined in Chapter 2 is right-continuous but not complete under each $\mathbb{P} \in \mathcal{P}_H$. However, as shown in Lemma* 2.4 *of [103], for every $\mathbb{P} \in \mathcal{P}_H$, we can always consider a version which is progressively measurable for the completion of $\mathbb{F}^+$ under $\mathbb{P}$. This shows that all the usual properties are still satisfied in our framework.*

3.3 Robust utility maximization

We will now present the main problem of this paper and introduce a financial market with volatility uncertainty. The financial market consists of one bond with zero interest rate and d stocks. The price process is given by

$$dS_t = \operatorname{diag}[S_t](b_t dt + dB_t),\ \mathcal{P}_H - q.s.$$

where b is an $\mathbb{R}^d$-valued uniformly bounded stochastic process which is uniformly continuous in ω for the $||\cdot||_\infty$ norm.

Remark 3.3.1. *The volatility is implicitly embedded in the model. Indeed, under each $\mathbb{P} \in \mathcal{P}_H$, we have $dB_s \equiv \widehat{a}_t^{1/2} dW_t^{\mathbb{P}}$ where $W^{\mathbb{P}}$ is a Brownian motion under $\mathbb{P}$. Therefore, $\widehat{a}^{1/2}$ plays the role of volatility under each $\mathbb{P}$ and thus allows us to model the volatility uncertainty. We also note that we make the uniform continuity assumption for b to ensure that the generators of the 2BSDEs obtained later satisfy Assumptions 2.2.1 or 2.2.2.*

We then denote $\pi = (\pi_t)_{0 \leqslant t \leqslant T}$ a trading strategy, which is a d-dimensional $\mathcal{F}$-progressively measurable process, supposed to take its value in some closed set A. We refer to Definitions 3.4.1, 3.5.1 and 3.6.1 in the following sections for precise definitions of the set of admissible strategies $\mathcal{A}$ for the three utility functions we study.

The process π_t^i describes the amount of money invested in stock i at time t, with $1 \leqslant i \leqslant d$. The number of shares is $\frac{\pi_t^i}{S_t^i}$. So the liquidation value of a trading strategy π with positive initial capital x is given by the following wealth process

$$X_t^\pi = x + \int_0^t \pi_s(dB_s + b_s ds),\ 0 \leqslant t \leqslant T,\ \mathcal{P}_H - q.s.$$

Since we assumed zero interest rate, the amount of money in the bank π^0 does not appear in the wealth process X.

Let ξ be a liability that matures at time T, which is a random variable assumed to be $\mathcal{F}_T$-measurable and in $\mathcal{L}_H^\infty$. The problem of the investor in this financial market is to maximize her expected utility under model uncertainty from her total wealth $X_T^\pi - \xi$. Let U be a utility function, then the value function V of the maximization problem can be written as

$$V^\xi(x) := \sup_{\pi \in \mathcal{A}} \inf_{\mathbb{Q} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{Q}}[U(X_T^\pi - \xi)]. \tag{3.3.1}$$

In the case where $\mathcal{P}_H$ contains only one probability measure, the problem reduces to the classical utility maximization problem.

Remark 3.3.2. *Due to the construction of 2BSDEs, we need the liability ξ to be in the class $\mathcal{L}_H^\infty$. It is easy to see that ξ can be constant, deterministic or in the form of $g(B_T)$ where g is a Lipschitz bounded function, such as a Put or a Call spread payoff function. However, we notice that vanilla options payoffs with underlying S may not be in $\mathcal{L}_H^\infty$. Indeed, we have in the one-dimensional framework*

$$S_T = S_0 \exp\left(\int_0^T b_t dt - \frac{1}{2}\langle B\rangle_T + B_T\right),\ \mathcal{P}_H - q.s.$$

Since the quadratic variation of the canonical process can be written as follows

$$\overline{\lim_{n\to+\infty}} \sum_{i \leqslant 2^n t} \left(B_{\frac{i+1}{2^n}}(\omega) - B_{\frac{i}{2^n}}(\omega) \right)^2,$$

it is not too difficult to see that S can be approximated by a sequence of random variables in $\mathrm{UC_b}(\Omega)$. Besides, this sequence converges in $\mathcal{L}^2_H$. However, we cannot be sure that it also converges in $\mathcal{L}^\infty_H$, which is our space of interest here.

Of course, in the uncertain volatility framework, this seems to be a major drawback. Nevertheless, to deal with these options, it suffices to redo the whole 2BSDE construction from scratch but taking the exponential of the Brownian motion under the Wiener measure as the canonical process instead of the Brownian motion itself. This would amount to restrict ourselves to the subset $\mathcal{P}^+_H$ of $\mathcal{P}_H$, containing only those $\mathbb{P} \in \mathcal{P}_H$ such that the canonical process is a positive continuous local martingale under $\mathbb{P}$.

To find the value function V^ξ and an optimal trading strategy π^*, we follow the ideas of the general *martingale optimality principle* approach as in [38] and [54], but adapt it here to a nonlinear framework. We recall that $\mathcal{A}$ is the admissibility set of the strategies π.

Let $\{R^\pi\}_{\pi\in\mathcal{A}}$ be a family of processes which satisfies the following properties

Properties 3.3.1. (i) $R^\pi_T = U(X^\pi_T - \xi)$ *for all* $\pi \in \mathcal{A}$.

(ii) $R^\pi_0 = R_0$ *is constant for all* $\pi \in \mathcal{A}$.

(iii) *We have*

$$R^\pi_t \geqslant \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[R^\pi_T], \ \forall \pi \in \mathcal{A}$$

$$R^{\pi^*}_t = \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[R^{\pi^*}_T] \text{ for some } \pi^* \in \mathcal{A},\ \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H.$$

Then it follows

$$\inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[U(X^\pi_T - \xi)] \leqslant R_0 = \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[U(X^{\pi^*}_T - \xi)] = V^\xi(x). \tag{3.3.2}$$

In the following sections we will follow the ideas of Hu, Imkeller and Müller [54] to construct such a family for our three utility functions U.

3.4 Robust exponential utility

In this section, we will consider the exponential utility function which is defined as

$$U(x) = -\exp(-\beta x),\ x \in \mathbb{R} \text{ for } \beta > 0.$$

In our context, the set of admissible trading strategies is defined as follows

Definition 3.4.1. *Let A be a closed set in $\mathbb{R}^d$. The set of admissible trading strategies $\mathcal{A}$ consists of all d-dimensional progressively measurable processes, $\pi = (\pi_t)_{0 \leqslant t \leqslant T}$ satisfying*

$$\pi \in \mathbb{B}\mathrm{MO}(\mathcal{P}_H) \text{ and } \pi_t \in A,\ dt \otimes \mathcal{P}_H - a.e.$$

Remark 3.4.1. *Many authors have shed light on the natural link between BMO class, exponential uniformly integrable class and BSDEs with quadratic growth. See [12], [6] and [54] among others. In the standard utility maximization problem studied in [54], their trading strategies satisfy a uniform integrability assumption on the family $(\exp(X^\pi_\tau))_\tau$. Since the optimal strategy is a BMO martingale, it is easy to see that the utility maximization problem can also be solved if the uniform integrability assumption is replaced by a BMO assumption. However, at the end of the day, those two assumptions are deeply linked, as shown in the context of quadratic semimartingales in [6]. Nonetheless, in our framework, as explained below in Remark 3.4.3, we need to generalize the BMO martingale assumption instead of the uniform integrability assumption. Moreover, as recalled in the Introduction, from a financial point of view these admissibility sets are related to absence of arbitrage in the market considered.*

3.4.1 Characterization of the value function and existence of an optimal strategy

The investor wants to solve the maximization problem

$$V^\xi(x) := \sup_{\pi \in \mathcal{A}} \inf_{\mathbb{Q} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{Q}}\left[-\exp\left(X^\pi_T - \xi\right)\right]. \tag{3.4.1}$$

In order to construct a process R^π which satisfies the Properties 3.3.1, we set

$$R^\pi_t = -\exp(-\beta(X^\pi_t - Y_t)),\ t \in [0,T],\ \pi \in \mathcal{A},$$

where $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is the unique solution of a 2BSDE with a well chosen quadratic generator $\widehat{F}$ satisfying Assumption 2.2.1 or 2.2.2

$$Y_t = \xi - \int_t^T Z_s dB_s - \int_t^T \widehat{F}(s, Z_s) ds + K^{\mathbb{P}}_T - K^{\mathbb{P}}_t,\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H.$$

Remark 3.4.2. *From Theorem 2.3.1 of Chapter 2, we have the following representation*

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})} y^{\mathbb{P}'}_t(T,\xi).$$

Therefore, in general Y_0 is only $\mathcal{F}_{0^+}$-measurable and therefore not a constant. But by Proposition 2.5.2 of Chapter 2, we know that the process Y is actually $\mathbb{F}$-measurable (this is true when the terminal condition is in $\mathrm{UC_b}(\Omega)$ and by passing to the limit when the terminal condition is in $\mathcal{L}^\infty_H$). This and the above representation imply easily that

$$Y_0 = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(0^+,\mathbb{P})} y^{\mathbb{P}'}_0(T,\xi) = \sup_{\mathbb{P}' \in \mathcal{P}_H} y^{\mathbb{P}'}_0(T,\xi),$$

then by the Blumenthal Zero-One law Y_0 is a constant.

Let us now define for all $a \in \mathbb{S}_d^{>0}$ such that $\underline{a} \leqslant a \leqslant \overline{a}$ the set A_a by

$$A_a := a^{1/2}A = \left\{a^{1/2}b, \ b \in A\right\}.$$

For any $a \in [\underline{a}, \overline{a}]$, the set A_a is still closed. Moreover, since $A \neq \varnothing$ we have

$$\min\left\{|r|\,, \ r \in A_a\right\} \leqslant k, \tag{3.4.2}$$

for some constant k independent of a.

We can now state the main result of this section

Theorem 3.4.1. *Assume that $\xi \in \mathcal{L}_H^\infty$ and either that $\|\xi\|_{\mathbb{L}_H^\infty} + \sup\limits_{0 \leqslant t \leqslant T} \|b_t\|_{\mathbb{L}_H^\infty}$ is small and that $0 \in A$, or that the set A is C^2 (in the sense that its border is a C^2 Jordan arc). Then, the value function of the optimization problem (3.4.1) is given by*

$$V^\xi(x) = -\exp\left(-\beta\left(x - Y_0\right)\right),$$

where Y_0 is defined as the initial value of the unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the following 2BSDE

$$Y_t = \xi - \int_t^T Z_s dB_s - \int_t^T \widehat{F}_s(Z_s)ds + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H. \tag{3.4.3}$$

The generator is defined as follows

$$\widehat{F}_t(\omega, z) := F_t(\omega, z, \widehat{a}_t), \tag{3.4.4}$$

where for all $t \in [0,T]$, $z \in \mathbb{R}^d$ and $a \in \mathbb{S}_d^{>0}$

$$F_t(\omega, z, a) = -\frac{\beta}{2}\mathrm{dist}^2\left(a^{1/2}z + \frac{1}{\beta}\theta_t(\omega), A_a\right) + z'a^{1/2}\theta_t(\omega) + \frac{1}{2\beta}\left|\theta_t(\omega)\right|^2,$$

where $\theta_t(\omega) := a^{-1/2}b_t(\omega)$ and where for any $x \in \mathbb{R}^d$ and any set $E \subset \mathbb{R}^d$, $\mathrm{dist}(x, E)$ denotes the distance from x to E.

Moreover, there exists an optimal trading strategy π^* satisfying

$$\widehat{a}_t^{1/2}\pi_t^* \in \Pi_{A_{\widehat{a}_t}}\left(\widehat{a}_t^{1/2}Z_t + \frac{1}{\beta}\widehat{\theta}_t\right), \quad t \in [0,T], \ \mathcal{P}_H - q.s. \tag{3.4.5}$$

with $\widehat{\theta}_t := \widehat{a}_t^{-1/2}b_t$.

Proof. The proof is divided into 5 steps. First, we show that the 2BSDE with the generator defined in (3.4.4) has indeed a unique solution. Then, we prove a multiplicative decomposition for the process R^π and some BMO integrability results on the process Z and the optimal strategy π^*. Using these results, we are then able to show that (iii) of Properties 3.3.1 holds.

Step 1: We first show that the 2BSDE (3.4.3) has an unique solution. We need to verify that the generator $\widehat{F}$ satisfies the conditions of Assumption 2.2.2 or 2.2.1.

First of all, F defined above is a convex function of a, for all $a \in \mathbb{S}_d^{>0}$, and thus for any $t \in [0,T]$, F can be written as the Fenchel transform of a function

$$H_t(\omega, z, \gamma) := \sup_{a \in D_F} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) - F_t(\omega, z, a) \right\} \text{ for } \gamma \in \mathbb{R}^{d \times d}.$$

That F satisfies the first two conditions of either Assumption 2.2.2 or 2.2.1 is obvious. For Assumptions 2.2.2(iii) and 2.2.1(iii), the assumption of boundedness and uniform continuity in ω on b implies that b^2 is uniformly continuous in ω. Since b and b^2 are the only non-deterministic terms in F, then F is also uniformly continuous in ω.

Then, since we consider the distance function to a closed set, we know that it is attained for some element of $\mathbb{R}^d$. It is therefore clear that the generator of this 2BSDE is linear and quadratic in z. Besides, as recalled earlier in (3.4.2), there exists a constant $k \geqslant 0$ such that

$$\min\left\{|d|\,,\ d \in A_{\widehat{a}_t}\right\} \leqslant k \quad \text{for } dt \otimes \mathbb{P} - a.e., \text{ for all } \mathbb{P} \in \mathcal{P}_H.$$

Then we get, for all $z \in \mathbb{R}^d$, $t \in [0,T]$,

$$\mathrm{dist}^2\left(\widehat{a}_t^{1/2} z + \frac{1}{\beta}\widehat{\theta}_t, A_{\widehat{a}_t}\right) \leqslant 2\left|\widehat{a}_t^{1/2} z\right|^2 + 2\left(\frac{1}{\beta}\left|\widehat{\theta}_t\right| + k\right)^2.$$

Thus, we obtain from the boundedness of $\widehat{\theta}$

$$\left|\widehat{F}_t(z)\right| \leqslant c_0 + c_1 \left|\widehat{a}_t^{1/2} z\right|^2,$$

that is to say that Assumptions 2.2.2(iv) and 2.2.1(iv) are satisfied.

Finally, Assumption 2.2.2(v) is clear from the Lipschitz property of the distance function, and Assumption 2.2.1(v) is also clear by our regularity assumption on the border of A.

The terminal condition ξ is in $\mathcal{L}_H^\infty$ and we have proved that the generator $\widehat{F}$ satisfies Assumption 2.2.2 or Assumption 2.2.1. Moreover, by the definition of the generator F, it is clear that if the process b has a small $\mathbb{L}_H^\infty$-norm and if $0 \in A$, then $\widehat{F}^0$ also has a small $\mathbb{L}_H^\infty$-norm. Indeed, in this case we have

$$\widehat{F}_t^0 = -\frac{\beta}{2}\mathrm{dist}\left(\frac{\theta_t}{\beta}, A_{\widehat{a}_t}\right) + \frac{1}{2\beta}\left|\theta_t\right|^2,$$

which tends to 0 as b_t and thus θ_t goes to 0 (this is clear for the second term on the right-hand side, and for the first one, continuity of the distance function and the fact $0 \in A$ ensure the result).

Therefore Theorem 2.5.1 in Chapter 2 states that the 2BSDE (3.4.3) has a unique solution in $\mathbb{D}_H^\infty \times \mathbb{H}_H^2$.

Step 2: We first decompose R^π as the product of a process M^π and a non-increasing process N^π that is constant for some $\pi^* \in \mathcal{A}$.

Define for all $\mathbb{P} \in \mathcal{P}_H$ any for any $t \in [0,T]$

$$M_t^\pi = e^{-\beta(x-Y_0)}\exp\left(-\int_0^t \beta(\pi_s - Z_s)dB_s - \frac{1}{2}\int_0^t \beta^2 \left|\widehat{a}_s^{1/2}(\pi_s - Z_s)\right|^2 ds - \beta K_t^{\mathbb{P}}\right),\ \mathbb{P}-a.s.$$

We can then write for all $t \in [0,T]$

$$R_t^\pi = M_t^\pi N_t^\pi,$$

with

$$N_t^\pi = -\exp\left(\int_0^t v(s,\pi_s,Z_s)ds\right),$$

and

$$v(t,\pi,z) \ = -\beta\pi b_t + \beta \widehat{F}_t(z) + \frac{1}{2}\beta^2\left|\widehat{a}_t^{1/2}(\pi - z)\right|^2.$$

Clearly, for every $t \in [0,T]$, we may rewrite $v(t,\pi_t,Z_t)$ in the following form

$$\begin{aligned}\frac{1}{\beta}v(t,\pi_t,Z_t) &= \frac{\beta}{2}\left|\widehat{a}_t^{1/2}\pi_t\right|^2 - \beta\pi_t'\widehat{a}_t^{1/2}\left(\widehat{a}_t^{1/2}Z_t + \frac{1}{\beta}\widehat{\theta}_t\right) + \frac{\beta}{2}\left|\widehat{a}_t^{1/2}Z_t\right|^2 + \widehat{F}_t(Z_t)\\ &= \frac{\beta}{2}\left|\widehat{a}_t^{1/2}\pi_t - \left(\widehat{a}_t^{1/2}Z_t + \frac{1}{\beta}\widehat{\theta}_t\right)\right|^2 - Z_t'\widehat{a}_t^{1/2}\widehat{\theta}_t - \frac{1}{2\beta}\left|\widehat{\theta}_t\right|^2 + \widehat{F}_t(Z_t).\end{aligned}$$

By a classical measurable selection theorem (see [31] or Lemma 3.1 in [33]), we can define a progressively measurable process π^* satisfying (3.4.5). Then, it follows from the definition of $\widehat{F}$ that $\mathcal{P}_H - q.s.$

- $v(t,\pi_t,Z_t) \geqslant 0$ for all $\pi \in \mathcal{A}$, $t \in [0,t]$.
- $v(t,\pi_t^*,Z_t) = 0$, $t \in [0,T]$,

which implies that the process N^π is always non-increasing for all π and is equal to -1 for π^*.

Step 3: In this step, we show that the processes

$$\int_0^\cdot Z_s dB_s,\quad \int_0^\cdot \pi_s^* dB_s,$$

are $\mathbb{B}\mathrm{MO}(\mathcal{P}_{\mathrm{H}})$ martingales.

First of all, by Lemma 2.2.1 in Chapter 2, we know that $\int_0^\cdot Z_s dB_s$ is a $\mathbb{B}\mathrm{MO}(\mathcal{P}_{\mathrm{H}})$ martingale. By the triangle inequality and the definition of π^* together with (3.4.2), we have for all $t \in [0,T]$

$$\begin{aligned}\left|\widehat{a}_t^{1/2}\pi_t^*\right| &\leqslant \left|\widehat{a}_t^{1/2}Z_t + \frac{1}{\beta}\widehat{\theta}_t\right| + \left|\widehat{a}_t^{1/2}\pi_t^* - \left(\widehat{a}_t^{1/2}Z_t + \frac{1}{\beta}\widehat{\theta}_t\right)\right|\\ &\leqslant 2\left|\widehat{a}_t^{1/2}Z_t\right| + \frac{2}{\beta}\left|\widehat{\theta}_t\right| + k \leqslant 2\left|\widehat{a}_t^{1/2}Z_t\right| + k_1,\end{aligned}$$

where k_1 is a bound on $\widehat{\theta}$.

Then, for every probability $\mathbb{P} \in \mathcal{P}_H$ and every stopping time $\tau \leqslant T$,

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T \left|\widehat{a}_t^{1/2}\pi_t^*\right|^2 dt\right] \leqslant \mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T 8\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt + 2Tk_1^2\right],$$

and therefore

$$\|\pi^*\|_{\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})} \leqslant 8\,\|Z\|_{\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})} + 2Tk_1^2.$$

This implies the $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ martingale property of $\int_0^\cdot \pi_s^* dB_s$ as desired.

Step 4: We then prove that $\pi^* \in \mathcal{A}$ and $R^{\pi^*} \equiv -M^{\pi^*}$ satisfies (iii) of Properties 3.3.1, that is to say for all $t \in [0,T]$

$$\underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[M_T^{\pi^*}\right] = M_t^{\pi^*},\quad \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H.$$

For a fixed $\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})$, we denote

$$L_t := \int_0^t \beta(\pi_s^* - Z_s)dB_s + \frac{1}{2}\int_0^t \beta^2\left|\widehat{a}_s^{1/2}(\pi_s^* - Z_s)\right|^2 ds + \beta K_t^{\mathbb{P}'},\ 0 \leqslant t \leqslant T,$$

then with Itô's formula, we obtain for every $t \in [0,T]$, thanks to the $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ property proved in Step 3

$$\begin{aligned}\mathbb{E}_t^{\mathbb{P}'}\left[M_T^{\pi^*}\right] - M_t^{\pi^*} &= -\beta\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*} dK_s^{\mathbb{P}'}\right] \\ &\quad + \mathbb{E}_t^{\mathbb{P}'}\left[\sum_{t \leqslant s \leqslant T} e^{-L_s} - e^{-L_{s^-}} + e^{-L_{s^-}}(L_s - L_{s^-})\right].\end{aligned} \tag{3.4.6}$$

First, we prove

$$\underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*} dK_s^{\mathbb{P}'}\right] = 0,\ t \in [0,T],\ \mathbb{P}-\text{a.s.}$$

For every t and every $\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})$, we have

$$0 \leqslant \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*} dK_s^{\mathbb{P}'}\right] \leqslant \mathbb{E}_t^{\mathbb{P}'}\left[\left(\sup_{0 \leqslant s \leqslant T} M_s^{\pi^*}\right)\left(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)\right].$$

Besides, since $K^{\mathbb{P}'}$ is nondecreasing, we obtain for all $s \geqslant t$

$$M_s^{\pi^*} \leqslant e^{-\beta(x - Y_0)}\mathcal{E}\left(\beta\int_0^s (Z_u - \pi_u^*)\,dB_u\right).$$

Then, again thanks to Step 3, we know that

$$(Z_s - \pi_s^*) \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}}),$$

and thus the exponential martingale above is a uniformly integrable martingale for all $\mathbb{P}$ and is in L^r_H for some $r > 1$ (see Lemma 2.2.2 in Chapter 2). Thus, by Hölder inequality, we have for all $t \in [0, T]$

$$\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s-}^{\pi^*} dK_s^{\mathbb{P}'}\right] \leqslant e^{\beta(Y_0-x)}\mathbb{E}_t^{\mathbb{P}'}\left[\sup_{0\leqslant s\leqslant T}\mathcal{E}^r\left(\beta\int_0^s (Z_u-\pi_u^*)\,dB_u\right)\right]^{\frac{1}{r}}\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^q\right]^{\frac{1}{q}}.$$

With Doob's maximal inequality, we have for every $t \in [0, T]$

$$\mathbb{E}_t^{\mathbb{P}'}\left[\sup_{0\leqslant s\leqslant T}\mathcal{E}^r\left(\beta\int_0^s (Z_u-\pi_u^*)\,dB_u\right)\right]^{1/r} \leqslant C\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}^r\left(\beta\int_0^T (Z_u-\pi_u^*)\,dB_u\right)\right]^{1/r} < +\infty,$$

where C is an universal constant that can change value from line to line.

Then by the Cauchy-Schwarz inequality, we get for $0 \leqslant t \leqslant T$

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^q\right]^{1/q} &\leqslant C\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\right]\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^{2q-1}\right]\right)^{\frac{1}{2q}} \\
&\leqslant C\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^{2q-1}\right]\right)^{\frac{1}{2q}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\right]\right)^{\frac{1}{2q}}.
\end{aligned}$$

Arguing as in the proof of Theorem 2.3.1 in Chapter 2 we know that

$$\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^{2q-1}\right]\right)^{\frac{1}{2q}} < +\infty,\ 0\leqslant t\leqslant T.$$

Hence, we obtain for $0 \leqslant t \leqslant T$

$$0 \leqslant \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s-}^{\pi^*} dK_s^{\mathbb{P}'}\right] \leqslant C\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\right]\right)^{\frac{1}{2q}} = 0,$$

which means

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s-}^{\pi^*} dK_s^{\mathbb{P}'}\right] = 0,\ 0\leqslant t\leqslant T.$$

Finally, we have for every $t \in [0, T]$

$$\begin{aligned}
&\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s-}^{\pi^*} dK_s^{\mathbb{P}'} - \sum_{t\leqslant s\leqslant T}\exp(-\beta L_s)-\exp(-\beta L_{s-})+\beta\exp(-\beta L_{s-})(L_s-L_{s-})\right] \\
&\leqslant \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s-}^{\pi^*} dK_s^{\mathbb{P}'}\right] \\
&\quad - \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\mathbb{E}_t^{\mathbb{P}'}\left[\sum_{t\leqslant s\leqslant T}\exp(-\beta L_s)-\exp(-\beta L_{s-})+\beta\exp(-\beta L_{s-})(L_s-L_{s-})\right] \\
&\leqslant 0,
\end{aligned}$$

because the function $x \to \exp(-x)$ is convex and the jumps of L are positive.

Hence, using (3.4.6), we have for every $t \in [0, T]$

$$\underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'} \left[M_T^{\pi^*} - M_t^{\pi^*} \right] \geqslant 0.$$

But by definition M^{π^*} is the product of a martingale and a positive non-increasing process and is therefore a supermartingale. This implies that for every $t \in [0, T]$

$$\underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'} \left[M_T^{\pi^*} - M_t^{\pi^*} \right] = 0.$$

Finally, π^* is an admissible strategy, R^{π^*} satisfies (iii) of Properties 3.3.1 and

$$\begin{aligned} R_0^{\pi^*} &= \inf_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}} \left[-\exp\left(-\beta \left(x + \int_0^T \pi_s^* \left(dB_s + \theta_s ds \right) - \xi \right) \right) \right] \\ &= -\exp\left(-\beta \left(x - Y_0 \right) \right). \end{aligned}$$

Step 5: Next we will show that for all $\pi \in \mathcal{A}$, R^π satisfies (iii) of Properties 3.3.1, that is, for every $t \in [0, T]$

$$\underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'} [-\exp(-\beta(X_T^\pi - \xi))] \leqslant R_t^\pi, \ \mathbb{P} - a.s.$$

Since $\pi \in \mathcal{A}$, the process

$$\int_0^\cdot \left(Z_s - \pi_s \right) dB_s,$$

is a $\mathbb{B}$MO($\mathcal{P}_\mathrm{H}$) martingale. Then the process

$$G^\pi = \exp\left(-\beta(x - Y_0) \right) \mathcal{E} \left(-\beta \int_0^\cdot \left(\pi_s - Z_s \right) dB_s \right),$$

is a uniformly integrable martingale under each $\mathbb{P} \in \mathcal{P}_H$.

As in the previous steps, we write R^π as $R^\pi = M^\pi N^\pi$, where N^π is a negative non-increasing process. We then have for $0 \leqslant s \leqslant t \leqslant T$

$$\begin{aligned} \underset{\mathbb{P}' \in \mathcal{P}_H(s+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_s^{\mathbb{P}'} \left[M_t^\pi N_t^\pi \right] &\leqslant \underset{\mathbb{P}' \in \mathcal{P}_H(s+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_s^{\mathbb{P}'} \left[M_t^\pi N_s^\pi \right], \ \mathbb{P} - a.s. \\ &= \underset{\mathbb{P}' \in \mathcal{P}_H(s+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_s^{\mathbb{P}'} \left[M_t^\pi \right] N_s^\pi, \ \mathbb{P} - a.s. \end{aligned}$$

because N^π is negative. By the same arguments as in Step 3 for M^{π^*}, we have for $0 \leqslant s \leqslant t \leqslant T$

$$\underset{\mathbb{P}' \in \mathcal{P}_H(s+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_s^{\mathbb{P}'} \left[M_t^\pi \right] = M_s^\pi, \ \mathbb{P} - a.s.$$

Therefore the following inequality holds for $0 \leqslant s \leqslant t \leqslant T$

$$\underset{\mathbb{P}' \in \mathcal{P}_H(s+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_s^{\mathbb{P}'} \left[R_t^\pi \right] \leqslant R_s^\pi, \ \mathbb{P} - a.s.$$

which ends the proof. $\square$

Remark 3.4.3. *We see here why it is essential in our context to have strong integrability assumptions on the trading strategies. Indeed, in the proof of the above property for M^{π^*}, the fact that the stochastic integral*

$$\int_0^{\cdot} \pi_s^* dB_s,$$

is a $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ martingale allowed us to control the moments of its stochastic exponential, which in turn allowed us to deduce from the minimal property for $K^{\mathbb{P}}$ a similar minimal property for

$$\int_0^{\cdot} M_s^{\pi^*} dK_s^{\mathbb{P}}.$$

This term is new when compared with the context of [54]. To deal with it, we have to impose the $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ property. Let us note however that since the optimal strategy already has this property, we do not lose much by restricting the strategies.

Remark 3.4.4. *We note that our approach still works when there are no constraints on trading strategies. In this case, the 2BSDE related to the maximization problem has a uniformly Lipschitz generator, thus the theory developed in [101] for Lipschitz 2BSDEs can be used.*

3.4.2 A min-max property

By comparing the value function of our robust utility maximization problem and the one presented in [54] for standard utility maximization problem, we are able to have a min-max property similar to the one obtained by Denis and Kervarec in [29]. We observe that we were only able to prove this property after having solved the initial problem, unlike in the approach of [29].

Theorem 3.4.2. *Under the previous assumptions on the probability measures set $\mathcal{P}_H$ and the admissible strategies set $\mathcal{A}$, the following min-max property holds.*

$$\sup_{\pi\in\mathcal{A}} \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right] = \inf_{\mathbb{P}\in\mathcal{P}_H} \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right] = \inf_{\mathbb{P}\in\mathcal{P}_H} \sup_{\pi\in\mathcal{A}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right],$$

where $\mathcal{A}^{\mathbb{P}}$ is the set consisting of trading strategies π which are in A and such that the process $\left(\int_0^t \pi_s dB_s\right)_{0\leqslant t\leqslant T}$ is a BMO($\mathbb{P}$) martingale.

Proof. First note that we have

$$D := \sup_{\pi\in\mathcal{A}} \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right] \leqslant \inf_{\mathbb{P}\in\mathcal{P}_H} \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right] \leqslant \inf_{\mathbb{P}\in\mathcal{P}_H} \sup_{\pi\in\mathcal{A}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right] =: C.$$

Indeed, the first inequality is obvious and the second one follows from the fact that for all $\mathbb{P}$, $\mathcal{A} \subset \mathcal{A}^{\mathbb{P}}$.

It remains to prove that $C \leqslant D$. By the previous sections, we know that

$$D = -\exp\left(-\beta\left(x - Y_0\right)\right).$$

Moreover, we know from Chapter 2 that we have a representation for Y_0,

$$Y_0 = \sup_{\mathbb{P}\in\mathcal{P}_H} y_0^{\mathbb{P}},$$

where $y_0^{\mathbb{P}}$ is the solution of the standard BSDE with the same generator $\widehat{F}$.

On the other hand, we observe from [54] that

$$C = \inf_{\mathbb{P}\in\mathcal{P}_H} \left[-\exp\left(-\beta\left(x - y_0^{\mathbb{P}}\right)\right)\right],$$

implying that $C = D$. □

3.4.3 Indifference pricing via robust utility maximization

It has been shown in [38] that in a market model with constraints on the portfolios, if we define the indifference price for a contingent claim Φ as the smallest number p such that

$$\sup_{\pi} \mathbb{E}\left[-\exp\left(-\beta\left(X^{x+p,\pi} - \Phi\right)\right)\right] \geqslant \sup_{\pi} \mathbb{E}\left[-\exp\left(-\beta X^{x,\pi}\right)\right],$$

where $X^{x,\pi}$ is the wealth associated with the portfolio π and initial value x, then this problem turns into the resolution of BSDEs with quadratic growth generators.

In our framework of uncertain volatility, the problem of indifference pricing of a contingent claim Φ boils down to solve the following equation in p

$$V^0(x) = V^{\Phi}(x+p).$$

Thanks to our results, we know that if $\Phi \in \mathcal{L}_H^{\infty}$ then the two sides of the above equality can be calculated by solving 2BSDEs. The price p can therefore be calculated as soon as we are able to solve the 2BSDEs (explicitly or numerically). We provide two examples in Section 3.7.

3.5 Robust power utility

In this section, we will consider the power utility function

$$U(x) = -\frac{1}{\gamma}x^{-\gamma},\ x > 0,\ \gamma > 0.$$

Here we shall use a different notion of trading strategy: $\rho = (\rho^i)_{i=1,\dots,d}$ denotes the proportion of wealth invested in stock i. The number of shares of stock i is given by $\frac{\rho_t^i X_t}{S_t^i}$.

Then the wealth process is defined as

$$X_t^{\rho} = x + \int_0^t \sum_{i=1}^{d} \frac{X_s^{\rho}\rho_s^i}{S_s^i} dS_s^i = x + \int_0^t X_s^{\rho}\rho_s\left(dB_s + b_s ds\right),\ \mathcal{P}_H - q.s. \tag{3.5.1}$$

and the initial capital x is positive.

In the present setting, the set of admissible strategies is defined as follows

Definition 3.5.1. *Let A be a closed set in $\mathbb{R}^d$. The set of admissible trading strategies $\mathcal{A}$ consists of all $\mathbb{R}^d$-valued progressively measurable processes $\rho = (\rho_t)_{0 \leqslant t \leqslant T}$ satisfying*

$$\rho \in \mathbb{BMO}(\mathcal{P}_H) \text{ and } \rho \in A,\ dt \otimes \mathcal{P}_H - a.e.$$

The wealth process X^ρ can be written as

$$X_t^\rho = x\mathcal{E}\left(\int_0^t \rho_s(dB_s + b_s ds)\right),\ t \in [0,T],\ \mathcal{P}_H - q.s.$$

Then for every $\rho \in \mathcal{A}$, the wealth process X^ρ is a local $\mathbb{P}$-martingale bounded from below, hence, a $\mathbb{P}$-supermartingale, for all $\mathbb{P} \in \mathcal{P}_H$.

We suppose that there is no liability ($\xi = 0$). Then the investor faces the maximization problem

$$V(x) = \sup_{\rho \in \mathcal{A}} \inf_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[U(X_T^\rho)\right]. \tag{3.5.2}$$

In order to find the value function and an optimal strategy, we apply the same method as in the exponential utility case. We therefore have to construct a stochastic process R^ρ with terminal value

$$R_T^\rho = U\left(x + \int_0^T X_s^\rho \rho_s \frac{dS_s}{S_s}\right).$$

satisfying Properties 3.3.1.

Then the value function will be given by $V(x) = R_0$. Applying the utility function to the wealth process yields

$$-\frac{1}{\gamma}\left(X_t^\rho\right)^{-\gamma} = -\frac{1}{\gamma}x^{-\gamma}\exp\left(-\int_0^t \gamma\rho_s dB_s - \int_0^t \gamma\rho_s b_s ds + \frac{1}{2}\int_0^t \gamma\left|\widehat{a}_s^{1/2}\rho_s\right|^2 ds\right). \tag{3.5.3}$$

This equation suggests the following choice

$$R_t^\rho = -\frac{1}{\gamma}x^{-\gamma}\exp\left(-\int_0^t \gamma\rho_s dB_s - \int_0^t \gamma\rho_s b_s ds + \frac{1}{2}\int_0^t \gamma\left|\widehat{a}_s^{1/2}\rho_s\right|^2 ds + Y_t\right),$$

where $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ is the unique solution of the following 2BSDE

$$Y_t = 0 - \int_t^T Z_s dB_s - \int_t^T \widehat{F}_s(Z_s)ds + K_T - K_t,\ t \in [0,T],\ \mathcal{P}_H - q.s. \tag{3.5.4}$$

In order to get (iii) of Properties 3.3.1 for R^ρ, we have to construct $\widehat{F}_t(z)$ such that, for $t \in [0,T]$

$$\gamma\rho_t b_t - \frac{1}{2}\gamma\left|\widehat{a}_t^{1/2}\rho_t\right|^2 - \widehat{F}_t(Z_t) \leqslant -\frac{1}{2}\left|\widehat{a}_t^{1/2}(\gamma\rho_t - Z_t)\right|^2 \text{ for all } \rho \in \mathcal{A}, \tag{3.5.5}$$

with equality for some $\rho^* \in \mathcal{A}$. This is equivalent to

$$\widehat{F}_t(Z_t) \geqslant -\frac{1}{2}\gamma(1+\gamma)\left|\widehat{a}_t^{1/2}\rho_t - \frac{1}{1+\gamma}\left(-\widehat{a}_t^{1/2}Z_t + \widehat{\theta}_t\right)\right|^2 - \frac{1}{2}\frac{\gamma\left|-\widehat{a}_t^{1/2}Z_t + \widehat{\theta}_t\right|^2}{1+\gamma} + \frac{1}{2}\left|\widehat{a}_t^{1/2}Z_t\right|^2.$$

Hence, the appropriate choice for $\widehat{F}$ is

$$\widehat{F}_t(z) = -\frac{\gamma(1+\gamma)}{2}\mathrm{dist}^2\left(\frac{-\widehat{a}_t^{1/2}z+\widehat{\theta}_t}{1+\gamma}, A_{\widehat{a}_t}\right) + \frac{\gamma\left|-\widehat{a}_t^{1/2}z+\widehat{\theta}_t\right|^2}{2(1+\gamma)} + \frac{1}{2}\left|\widehat{a}_t^{1/2}z\right|^2, \quad (3.5.6)$$

and a candidate for the optimal strategy must satisfy

$$\widehat{a}_t^{1/2}\rho_t^* \in \Pi_{A_{\widehat{a}_t}}\left(\frac{1}{1+\gamma}\left(-\widehat{a}_t^{1/2}Z_t+\widehat{\theta}_t\right)\right),\ t\in[0,T].$$

We summarize the above results in the following Theorem.

Theorem 3.5.1. *Assume either that the drift b verifies that $\sup_{0\leqslant t\leqslant T}\|b_t\|_{\mathbb{L}^\infty_H}$ is small and that the set A contains 0, or that the set A is C^2 (in the sense that its border is a C^2 Jordan arc). Then, the value function of the optimization problem (3.5.2) is given by*

$$V(x) = -\frac{1}{\gamma}x^{-\gamma}\exp(Y_0)\quad \text{for } x>0,$$

where Y_0 is defined as the initial value of the unique solution $(Y,Z)\in\mathbb{D}^\infty_H\times\mathbb{H}^2_H$ of the quadratic 2BSDE

$$Y_t = 0 - \int_t^T Z_s dB_s - \int_t^T \widehat{F}_s(Z_s)ds + K_T - K_t,\ t\in[0,T]\ \mathcal{P}_H - q.s. \quad (3.5.7)$$

where $\widehat{F}$ is given by (3.5.6).

Moreover, there exists an optimal trading strategy $\rho^*\in\mathcal{A}$ with the property

$$\widehat{a}_t^{1/2}\rho_t^* \in \Pi_{A_{\widehat{a}_t}}\left(\frac{1}{1+\gamma}\left(-\widehat{a}_t^{1/2}Z_t+\widehat{\theta}_t\right)\right),\ t\in[0,T]. \quad (3.5.8)$$

Proof. The proof is very similar to the case of robust exponential utility. First we can show, with the same arguments, that the generator $\widehat{F}$ satisfies the conditions of Assumption 2.2.1 or Assumption 2.2.2, hence there exists a unique solution to the 2BSDE (3.5.7).

Let then ρ^* denote the progressively measurable process, constructed with a measurable selection theorem, which realizes the distance in the definition of $\widehat{F}$. The same arguments as in the case of robust exponential utility show that $\rho^*\in\mathcal{A}$.

Then with the choice we made for $\widehat{F}$, we have the following multiplicative decomposition

$$R_t^\rho = -\frac{1}{\gamma}x^{-\gamma}\mathcal{E}\left(-\int_0^t(\gamma\rho_s - Z_s)\,dB_s\right)e^{-\gamma K_t^{\mathbb{P}}}\exp\left(-\int_0^t v_s ds\right),$$

where

$$v_t = \gamma\rho_t b_t - \frac{1}{2}\gamma\left|\widehat{a}_t^{1/2}\rho_t\right|^2 - \widehat{F}_t(Z_t) + \frac{1}{2}\left|\widehat{a}_t^{1/2}(\gamma\rho_t - Z_t)\right|^2 \leqslant 0,\ dt\otimes\mathbb{P}\text{−a.e.}$$

Then since the stochastic integral $\int_0^t (\rho_s - Z_s) dB_s$ is a $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ martingale, the stochastic exponential above is a uniformly integrable martingale. By exactly the same arguments as before, we have

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(s+,\mathbb{P})} \mathbb{E}_s^{\mathbb{P}'} \left[R_t^\rho \right] \leqslant R_s^\rho, \quad s \leqslant t, \ \mathbb{P} - a.s.$$

with equality for ρ^*.

Hence, the terminal value R_T^ρ is the utility of the terminal wealth of the trading strategy ρ. Consequently,

$$\inf_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}} \left[U \left(X_T^\rho \right) \right] \leqslant R_0 = -\frac{1}{\gamma} x^{-\gamma} \exp(Y_0) \ \text{ for all } \rho \in \mathcal{A}.$$

$\square$

Remark 3.5.1. *Of course, the min-max property of Theorem 3.4.2 still holds.*

3.6 Robust logarithmic utility

In this section, we consider logarithmic utility function

$$U(x) = \log(x), \ x > 0.$$

Here we use the same notion of trading strategies as in the power utility case, $\rho = (\rho^i)_{i=1,\dots,d}$ denotes the part of the wealth invested in stock i. The number of shares of stock i is given by $\frac{\rho_t^i X_t}{S_t^i}$. Then the wealth process is defined as

$$X_t^\rho = x + \int_0^t \sum_{i=1}^d \frac{X_s^\rho \rho_s^i}{S_s^i} dS_s^i = x + \int_0^t X_s^\rho \rho_s \left(dB_s + b_s ds \right), \ \mathcal{P}_H - q.s. \tag{3.6.1}$$

and the initial capital x is positive.

The wealth process X^ρ can be written as

$$X_t^\rho = x \mathcal{E} \left(\int_0^t \rho_s (dB_s + b_s ds) \right), \ t \in [0, T], \ \mathcal{P}_H - q.s.$$

In this case, the set of admissible strategies is defined as follows

Definition 3.6.1. *Let A be a closed set in $\mathbb{R}^d$. The set of admissible trading strategies $\mathcal{A}$ consists of all $\mathbb{R}^d$-valued progressively measurable processes ρ satisfying*

$$\sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}} \left[\int_0^T |\widehat{a}_t^{1/2} \rho_t|^2 dt \right] < \infty,$$

and $\rho \in A$, $dt \otimes d\mathbb{P} - a.s.$, $\forall \mathbb{P} \in \mathcal{P}_H$.

For the logarithmic utility, we assume the agent has no liability at time T ($\xi = 0$). Then the optimization problem is given by

$$\begin{aligned} V(x) &= \sup_{\rho\in\mathcal{A}} \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[\log(X_T^\rho)] \\ &= \log(x) + \sup_{\rho\in\mathcal{A}} \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \rho_s dB_s + \int_0^T (\rho_s b_s - \frac{1}{2}|\widehat{a}_s^{1/2}\rho_s|^2)ds\right]. \end{aligned} \tag{3.6.2}$$

We have the following theorem.

Theorem 3.6.1. *Assume either that the drift b verifies that $\sup_{0\leqslant t\leqslant T} \|b_t\|_{\mathbb{L}_H^\infty}$ is small and that the set A contains 0, or that the set A is C^2 (in the sense that its border is a C^2 Jordan arc). Then, the value function of the optimization problem (3.6.2) is given by*

$$V(x) = \log(x) - Y_0 \quad \text{for } x > 0,$$

where Y_0 is defined as the initial value of the unique solution $(Y,Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the quadratic 2BSDE

$$Y_t = 0 - \int_t^T Z_s dB_s - \int_t^T \widehat{F}_s ds + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ t\in[0,T],\ \mathbb{P}-a.s.,\ \forall \mathbb{P}\in\mathcal{P}_H. \tag{3.6.3}$$

The generator is defined by

$$\widehat{F}_s = F_s(\widehat{a}_s),$$

where

$$F_s(a) = -\frac{1}{2}\text{dist}^2(\theta_s, A_a) + \frac{1}{2}|\theta_s|^2,\ \textit{for } a\in\mathbb{S}_d^{>0}.$$

Moreover, there exists an optimal trading strategy $\rho^* \in \mathcal{A}$ with the property

$$\widehat{a}_t^{1/2}\rho_t^* \in \Pi_{A_{\widehat{a}_t}}\left(\widehat{\theta}_t\right). \tag{3.6.4}$$

Proof. The proof is very similar to the case of exponential and power utility. First we show that there exists an unique solution to the 2BSDE (3.6.3). We then write, for $t\in[0,T]$

$$R_t^\rho = M_t^\rho + N_t^\rho,$$

where

$$\begin{aligned} M_t^\rho &= \log(x) - Y_0 + \int_0^t (\rho_s - Z_s)\, dB_s + K_t^{\mathbb{P}}, \\ N_t^\rho &= \int_0^t \left(-\frac{1}{2}\left|\widehat{a}_s^{1/2}\rho_s - \widehat{\theta}_s\right|^2 + \frac{1}{2}\left|\widehat{\theta}_s\right|^2 - \widehat{F}_s\right) ds. \end{aligned}$$

Then, we similarly prove that ρ^*, which can be constructed by means of a classical measurable selection argument, is in $\mathcal{A}$. Note in particular that ρ^* only depends on $\widehat{\theta}$, $\widehat{a}^{1/2}$ and the closed set A describing the constraints on the trading strategies.

Next, due to Definition 3.6.1, the stochastic integral in R^ρ is a martingale under each $\mathbb{P}$ for all $\rho \in \mathcal{A}$. Moreover, $\widehat{F}$ is chosen to make the process N^ρ non-increasing for all ρ and a constant for ρ^*. Thus, the minimum condition of $K^{\mathbb{P}}$ implies that R^ρ satisfies (iii) of Properties 3.3.1.

Furthermore, the initial value Y_0 of the simple 2BSDE (3.6.3) satisfies

$$Y_0 = \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[-\int_0^T \widehat{F}_s ds\right].$$

Hence,

$$V(x) = R_0^{\rho^*}(x) = \log(x) - \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[-\int_0^T \widehat{F}_s ds\right].$$

□

Remark 3.6.1. *Of course, the min-max property of Theorem 3.4.2 still holds. Moreover, it is an easy exercise to show that the 2BSDE* (3.6.3) *has a unique solution given by*

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[\int_t^T \frac{1}{2}\left(\mathrm{dist}^2(\theta_s, A_{\widehat{a}_s}) - |\theta_s|^2\right) ds\right].$$

3.7 Examples

In general, it is difficult to solve BSDEs and 2BSDEs explicitly. In this section, we will give some examples where we have an explicit solution. In particular, we show how the optimal probability measure is chosen. In all our examples, we will work in dimension one, $d = 1$.

First, we deal with robust exponential utility. We consider the case where there are no constraints on trading strategies, that is $A = \mathbb{R}$. Then the associated 2BSDE has a generator which is linear in z. In the first example, we consider a deterministic terminal liability ξ and show that we can compare our result with the one obtained by solving the HJB equation in the standard Merton's approach, working with the probability measure associated to the constant process $\overline{a}$. In the second example, we show that with a random payoff $\xi = -B_T^2$, where B is the canonical process, we end up with an optimal probability measure which is not of Bang-Bang type (Bang-Bang type means that, under this probability measure, the density of the quadratic variation $\widehat{a}$ takes only the two extreme values, $\underline{a}$ and $\overline{a}$). We emphasize that this example does not have real financial significance, but shows nonetheless that one cannot expect the optimal probability measure to depend only on the two bounds for the volatility unlike with option pricing in the uncertain volatility model.

3.7.1 Example 1: Deterministic payoff

In this example, we suppose that b is a constant in $\mathbb{R}$. From Theorem 3.4.1, we know that the value function of the robust maximization problem is given by

$$V^\xi(x) = -\exp\left(-\beta\left(x - Y_0\right)\right),$$

where Y is the solution of a 2BSDE with a quadratic generator. When there are no constraints, the 2BSDE can be written as follows

$$Y_t = \xi - \int_t^T Z_s dB_s - \int_t^T \widehat{F}_s(Z_s)ds + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H.$$

and the generator is given by

$$\widehat{F}_t(z) := F_t(\omega, z, \widehat{a}) = bz + \frac{b^2}{2\beta\widehat{a}}.$$

Then we can solve explicitly the corresponding BSDEs with the same generator under each $\mathbb{P}$. Let

$$M_t = e^{-\int_0^t \frac{1}{2} b^2 \widehat{a}_s^{-1} ds - \int_0^t b \widehat{a}_s^{-1} dB_s}.$$

By applying Itô's formula to $y_t^{\mathbb{P}} M_t$, we have

$$y_0^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}\left[\xi M_T - \frac{b^2}{2\beta}\int_0^T \widehat{a}_s^{-1} M_s ds\right].$$

Since $\underline{a} \leqslant \widehat{a} \leqslant \overline{a}$, we derive that

$$y_0^{\mathbb{P}} \leqslant \xi - \frac{1}{2\beta}\frac{b^2}{\overline{a}}T.$$

Therefore, by the representation of Y, we have

$$Y_0 \leqslant \xi - \frac{1}{2\beta}\frac{b^2}{\overline{a}}T.$$

Moreover, under the specific probability measure $\mathbb{P}^{\overline{a}} \in \mathcal{P}_H$, we have

$$y_0^{\mathbb{P}^{\overline{a}}} = \xi - \frac{1}{2\beta}\frac{b^2}{\overline{a}}T.$$

This implies that $Y_0 = y_0^{\mathbb{P}^{\overline{a}}}$, which means that the robust utility maximization problem is degenerated and is equivalent to a standard utility maximization problem under the probability measure $\mathbb{P}^{\overline{a}}$. We discuss in more detail this result in Example 3.7.3 below.

3.7.2 Example 2 : Non-deterministic payoff

In this subsection, we consider a non-deterministic payoff $\xi = -B_T^2$. As in the first example, there are no constraints on trading strategies. Then, the 2BSDE has a linear generator. We can verify that $-B_T^2$ can be written as the limit under the norm $\|\cdot\|_{\mathbb{L}_H^2}$ of a sequence which is in $\mathrm{UC_b}(\Omega)$, and thus is in $\mathcal{L}_H^2$, which is the terminal condition set for 2BSDEs with Lipschitz generators. Here, we suppose that b is a deterministic continuous function of time t.

By the same method as in the previous example, let

$$M_t = e^{-\int_0^t \frac{1}{2} b_s^2 \widehat{a}_s^{-1} ds - \int_0^t b_s \widehat{a}_s^{-1} dB_s},$$

then we obtain

$$y_0^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}\left[-M_T B_T^2 - \int_0^T \frac{b_s^2}{2\beta}\widehat{a}_s^{-1} M_s ds\right].$$

By applying Itô's formula to $M_t B_t$, we have

$$dM_t B_t = M_t dB_t + B_t dM_t - b_t M_t dt.$$

Since b is deterministic, by taking expectation under $\mathbb{P}$ and localizing if necessary, we obtain

$$\mathbb{E}^{\mathbb{P}}\left[M_T B_T\right] = \mathbb{E}^{\mathbb{P}}\left[-\int_0^T b_t M_t dt\right] = -\int_0^T b_t dt.$$

Again, by applying Itô's formula to $-M_t B_t^2$, we have

$$-dM_t B_t^2 = -2M_t B_t dB_t - B_t^2 dM_t - \widehat{a}_t M_t dt + 2b_t M_t B_t dt.$$

Therefore $y_0^{\mathbb{P}}$ can be rewritten as

$$y_0^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}\left[\int_0^T -M_t\left(\widehat{a}_t + \frac{b_t^2}{2\beta\widehat{a}_t}\right)dt\right] - \int_0^T 2b_t\left(\int_0^t b_s ds\right)dt.$$

By analyzing the map $g : x \in \mathbb{R}^+ \longmapsto x - \frac{b_t^2}{2\beta x}$, we know that $g'(x) = 1 - \frac{b_t^2}{2\beta x^2}$, implying that g is nondecreasing when $x^2 \geqslant \frac{b_t^2}{2\beta}$.

Let us now assume that b is a deterministic positive continuous and nondecreasing function of time t such that

$$\frac{b_0^2}{2\beta} \leqslant \underline{a}^2 \leqslant \overline{a}^2 \leqslant \frac{b_T^2}{2\beta}.$$

Let $\underline{t}$ be such that $\frac{b_{\underline{t}}^2}{2\beta} = \underline{a}$ and $\overline{t}$ be such that $\frac{b_{\overline{t}}^2}{2\beta} = \overline{a}$, and define

$$a_t^* := \underline{a}\mathbf{1}_{0 \,\leqslant\, t \,\leqslant\, \underline{t}} + \frac{b_t}{\sqrt{2\beta}}\mathbf{1}_{\underline{t} \,\leqslant\, t \,\leqslant\, \overline{t}} + \overline{a}\mathbf{1}_{\overline{t} \,\leqslant\, t \,\leqslant\, T},\ 0 \leqslant t \leqslant T,$$

then as in Example 3.7.1, we can show that $\mathbb{P}^{a^*}$ is an optimal probability measure, which is not of Bang-Bang type.

3.7.3 Example 3 : Merton's approach for robust power utility

Here, we deal with robust power utility. As in Example 3.7.1, we suppose that b is a constant in $\mathbb{R}$ and $\xi = 0$. First, we consider the case where $A = \mathbb{R}$. From Theorem 3.5.1, $\widehat{F}_t(z)$ can be rewritten as

$$\widehat{F}_t(z) = \frac{\gamma\left|-\widehat{a}_t^{1/2}z + b\widehat{a}_t^{-1/2}\right|^2}{2(1+\gamma)} + \frac{1}{2}\left|\widehat{a}_t^{1/2}z\right|^2,$$

which is quadratic and linear in z.

According to BSDEs theory, we can solve explicitly the corresponding BSDEs with this generator under each probability measure $\mathbb{P}$. We use an exponential transformation and let

$$\alpha := 1 + \frac{\gamma}{1+\gamma}, \; y'^{\mathbb{P}} := e^{-\alpha y^{\mathbb{P}}}, \; z'^{\mathbb{P}} := e^{-\alpha y^{\mathbb{P}}} z^{\mathbb{P}}.$$

By applying Itô's formula, we know that $(y'^{\mathbb{P}}, z'^{\mathbb{P}})$ is the solution of the following linear BSDE

$$dy_t'^{\mathbb{P}} = -\alpha y_t'^{\mathbb{P}} \left[\frac{\gamma}{2(1+\gamma)} \left(b^2 \widehat{a}_t^{-1} - 2bz_t^{\mathbb{P}} \right) dt + z_t'^{\mathbb{P}} dB_t \right],$$

with the terminal condition $y_T'^{\mathbb{P}} = 1$.

For $t \in [0,T]$, let

$$\lambda_t := \frac{\alpha\gamma}{2(1+\gamma)} b^2 \widehat{a}_t^{-1}, \; \eta_t := -\frac{\gamma}{2(1+\gamma)} 2b \widehat{a}_t^{-1/2}, \text{ and } M_t := e^{\int_0^t \lambda_s - \frac{\eta_s^2}{2} ds + \int_0^t \widehat{a}_s^{-1/2} \eta_s dB_s}.$$

By applying Itô's formula to $y_t'^{\mathbb{P}} M_t$, we obtain

$$y_t'^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{P}} \left[M_T / M_t \right], \text{ so } y_0^{\mathbb{P}} = -\frac{1}{\alpha} \ln \left(\mathbb{E}^{\mathbb{P}} \left[M_T \right] \right).$$

Since $\underline{a} \leqslant \widehat{a} \leqslant \overline{a}$, we derive that

$$y_0^{\mathbb{P}} \leqslant -\frac{\gamma}{2(1+\gamma)} \frac{b^2}{\overline{a}} T.$$

Thus by the representation of Y, we have

$$Y_0 \leqslant -\frac{\gamma}{2(1+\gamma)} \frac{b^2}{\overline{a}} T.$$

Moreover, under the specific probability measure $\mathbb{P}^{\overline{a}} \in \mathcal{P}_H$, we have

$$y_0^{\mathbb{P}^{\overline{a}}} = -\frac{\gamma}{2(1+\gamma)} \frac{b^2}{\overline{a}} T.$$

This implies that $Y_0 = y_0^{\mathbb{P}^{\overline{a}}}$. Finally, the value of the robust power utility maximization problem is

$$V(x) = -\frac{1}{\gamma} x^{-\gamma} \exp\left(Y_0 \right).$$

As in Example 3.7.1, the robust utility maximization problem is degenerate, and becomes a standard utility maximization problem under the probability measure $\mathbb{P}^{\overline{a}}$. In order to shed more light on this somehow surprising result, we first recall the HJB equation obtained by Merton [81] in the standard utility maximization problem

$$-\frac{\partial v}{\partial t} - \sup_{\delta \in A} \left[\mathcal{L}^{\delta,\alpha} v(t,x) \right] = 0,$$

together with the terminal condition

$$v(T,x) = U(x) := -\frac{x^{-\gamma}}{\gamma}, \; x \in \mathbb{R}_+, \; \gamma > 0,$$

where

$$\mathcal{L}^{\delta,\alpha}v(t,x) = x\delta b\frac{\partial v}{\partial x} + \frac{1}{2}x^2\delta^2\alpha\frac{\partial^2 v}{\partial x^2},$$

with a constant volatility $\alpha^{1/2}$.

It turns out that, when $A = \mathbb{R}$, the value function is given by

$$v(t,x) = \exp\left(\frac{b^2}{2\alpha}\frac{-\gamma}{(1+\gamma)}(T-t)\right)U(x),\ \ (t,x) \in [0,T]\times\mathbb{R}_+.$$

Let $\alpha = \overline{a}$, we have $v(0,x) = V(x)$, which is the result given by our 2BSDE method. Intuitively and formally speaking (in the case of controls taking values in compact sets, it has actually been proved under other technical conditions in [105] that the solution to the stochastic game we consider is indeed a viscosity solution of the equation below, see also Remark 3.7.2), the HJB equation for the robust maximization problem should then be

$$-\frac{\partial v}{\partial t} - \sup_{\delta\in A}\inf_{\alpha\in[\underline{a},\overline{a}]}\left[\mathcal{L}^{\delta,\alpha}v(t,x)\right] = 0$$

together with the terminal condition $v(T,x) = U(x),\ \ x \in \mathbb{R}_+$.

Note that the value function we obtained from our 2BSDE approach solves the above PDE, confirming the intuition that this is the correct PDE to consider in this context. Now assume that $A = \mathbb{R}$. If the second derivative of v is positive, then the term

$$\sup_{\delta\in A}\inf_{\alpha\in[\underline{a},\overline{a}]}\left[\mathcal{L}^{\delta,\alpha}v(t,x)\right],$$

becomes infinite, so the above PDE has no meaning. This implies that v should be concave. Then $\overline{a}$ is the minimizer. This explains why the robust utility maximization problem degenerates in the case $A = \mathbb{R}$. From a financial point of view, this is the same type of results as in the problem of superreplication of an option with convex payoff under volatility uncertainty. Then, similarly as the so-called robustness of the Black-Scholes formula, this leads to the fact that the probability measure with the highest volatility corresponds to the worst-case for the investor. However, it is clear that when, for instance, we impose no short-sale and no large sales constraints (that is to say A is a segment), the problem should not degenerate and the optimal probability measure switches between the two bounds $\underline{a}$ and $\overline{a}$.

Finally, notice that using the language of G-expectation introduced by Peng in [89], if we let

$$G(\Gamma) = \frac{1}{2}\sup_{\underline{a}\leqslant\alpha\leqslant\overline{a}}\alpha\Gamma = \frac{1}{2}\left(\overline{a}\,(\Gamma)^+ - \underline{a}\,(\Gamma)^-\right),$$

then the above PDE can be rewritten as follows

$$-\frac{\partial v}{\partial t} + \inf_{\delta\in A}\left[\mathcal{L}^{\delta,\underline{a},\overline{a}}v(t,x)\right] = 0, \tag{3.7.1}$$

where

$$\mathcal{L}^{\delta,\underline{a},\overline{a}}v(t,x) = x^2\delta^2 G\left(-\frac{\partial^2 v}{\partial x^2}\right).$$

Then, our PDE plays the same role for Merton's PDE as the Black-Scholes-Barenblatt PDE plays for the usual Black-Scholes PDE, by replacing the second derivative terms by their non-linear versions.

Remark 3.7.1. *It could be interesting to consider more general constraints for the volatility process. For instance, we may hope to consider cases where $\underline{a}$ can become 0 and $\overline{a}$ can become $+\infty$. From the point of view of existence and uniqueness of the 2BSDEs with quadratic growth considered here, this is not a problem, since there is no uniform bound on $\widehat{a}$ for the set of probability measures considered in Chapter 2 (see Definition 2.2.1). However, this boundedness assumption is crucial to retain the BMO integrability of the optimal strategy and thus also crucial for our proofs. We think that without it, the problem could still be solved but by now using the dynamic programming and PDE approach that we mentioned. However, delicate problems would arise in the sense that on the one hand, if $\underline{a} = 0$, then the PDE will become degenerate and one should then have to consider solutions in the viscosity sense, and on the other hand, if $\overline{a} = +\infty$, the PDE will have to be understood in the sense of boundary layers.*

Another possible generalizations would be to consider time-dependent or stochastic uncertainty sets for the volatility. This would be possible if we were able to weaken Assumption 2.2.1(i)*, which was already crucial in the proofs of existence and uniqueness in [101]. One first step in this direction has been taken by Nutz in [86] where he defines a notion of G-expectation (which roughly corresponds to a 2BSDE with a generator equal to* 0*) with a stochastic domain of volatility uncertainty.*

Remark 3.7.2. *In [108], a similar problem of robust utility maximization is considered. They consider a financial market consisting of a riskless asset, a risky asset with unknown drift and volatility and an untradable asset with known coefficients. Their aim is to solve the robust utility maximization problem without terminal liability and without constraints for exponential and power utilities, by means of the dynamic programming approach already used in [105]. They managed to show that the value function of their problem solves a PDE similar to* (3.7.1)*, and also that (see Proposition* 2.2*) the optimal probability measure was of Bang-Bang type, thus confirming our intuition in their particular framework. Besides, they give some semi-explicit characterization of the optimal strategies and of the optimal probability measures. From a technical point of view, the main difference between our two approaches, beyond the methodology used, is that their set of generalized controls (that is to say their set of probability measures) is compact for the weak topology, because it corresponds to the larger set $\overline{\mathcal{P}}_W$ defined in Section 2.2 of Chapter 2. This is also the framework adopted in [29]. However, as shown in [27] for instance, our smaller set $\mathcal{P}_H$ is only relatively compact for the weak topology. Nonetheless, working with this smaller set has no effect from the point of view of applications, and more importantly allows us to obtain results which are not attainable by their PDE methods, for instance with non-Markovian terminal liability ξ and also when the set of trading strategies is constrained in an arbitrary closed set.*

Chapitre 4

Second Order Reflected BSDEs

4.1 Introduction

In this chapter, we study a class of 2RBSDEs with a given lower càdlàg obstacle. The outline is as follows. In Section 4.2, we provide the precise definition of 2RBSDEs and show how they are connected to classical RBSDEs. Next, in Section 4.3, we prove a representation formula for the Y-part of a solution of a 2RBSDE which in turn implies uniqueness. We then provide some links between 2RBSDEs and optimal stopping problems. In Section 4.4, we give a proof of existence by means of regular conditional probability distribution techniques, as in [101] for Lipschitz 2BDSEs. Let us mention that this proof requires to extend existing results on the theory of g-martingales of Peng (see [88]) to the reflected case. Since to the best of our knowledge, those results do not exist in the literature, we prove them in the Appendix 4.6. Finally, we use these new objects in Section 4.5 to study the pricing problem of American contingent claims in a market with volatility uncertainty. This chapter is based on [79].

4.2 Preliminaries

We consider the same framework as in Chapter 2 (see Section 2.2).

4.2.1 The nonlinear generator

Given a map $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a subset containing 0, we define the corresponding conjugate of H w.r.t.γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma\in D_H} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}_d^{>0},$$
$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0, 0).$$

We denote by $D_{F_t(y,z)} := \{a,\ F_t(\omega, y, z, a) < +\infty\}$ the domain of F in a for a fixed (t, ω, y, z).

As in [101] we fix a constant $\kappa \in (1, 2]$ and restrict the probability measures in $\mathcal{P}_H^\kappa \subset \overline{\mathcal{P}}_S$

Definition 4.2.1. *$\mathcal{P}_H^\kappa$ consists of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a}_{\mathbb{P}} \leqslant \widehat{a} \leqslant \bar{a}_{\mathbb{P}},\ dt \times d\mathbb{P} - a.s. \text{ for some } \underline{a}_{\mathbb{P}}, \bar{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0},\ \text{and } \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}_t^0\right|^\kappa dt\right)^{\frac{2}{\kappa}}\right] < +\infty$$

Definition 4.2.2. *We say that a property holds $\mathcal{P}_H^\kappa$-quasi-surely ($\mathcal{P}_H^\kappa$-q.s. for short) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H^\kappa$.*

We now state our main assumptions on the function F which will be our main interest in the sequel

Assumption 4.2.1. (i) *$\mathcal{P}_H^\kappa$ is not empty, and the domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *F is $\mathbb{F}$-progressively measurable in D_{F_t}.*

(iii) *We have the following uniform Lipschitz-type property in y and z*

$$\left|\widehat{F}_t(y,z) - \widehat{F}_t(y^{'},z^{'})\right| \leqslant C\left(\left|y-y^{'}\right| + \left|\widehat{a}^{1/2}\left(z-z^{'}\right)\right|\right),\ \mathcal{P}_H^\kappa - q.s.$$

for all $(t, y, y^{'}, z, z^{'})$.

(iv) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

Remark 4.2.1. *The assumptions (i) and (ii) are classic in the second order framework ([101]). The Lipschitz assumption (iii) is standard in the BSDE theory since the paper [87]. The last hypothesis (iv) is also proper to the second order framework, it is linked to our intensive use of regular conditional probability distributions (r.c.p.d.) in our existence proof, and to the fact that we construct our solutions pathwise, thus avoiding complex issues related to negligeable sets.*

Remark 4.2.2. (i) *$\mathcal{P}_H^\kappa$ is decreasing in κ since for $\kappa_1 < \kappa_2$ with Hölder's inequality*

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}_t^0\right|^{\kappa_1} dt\right)^{\frac{2}{\kappa_1}}\right] \leqslant C\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}_t^0\right|^{\kappa_2} dt\right)^{\frac{2}{\kappa_2}}\right].$$

(ii) *The Assumption 4.2.1, together with the fact that $\widehat{F}_t^0 < +\infty$, $\mathbb{P}$-a.s for every $\mathbb{P} \in \mathcal{P}_H^\kappa$, implies that $\widehat{a}_t \in D_{F_t}$, $dt \times \mathbb{P}$-a.s., for all $\mathbb{P} \in \mathcal{P}_H^\kappa$.*

4.2.2 The spaces and norms

We now recall from [101] the spaces and norms which will be needed for the formulation of the 2RBSDEs. Notice that all subsequent notations extend to the case $\kappa = 1$.

For $p \geqslant 1$, $L_H^{p,\kappa}$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|_{L_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty.$$

$\mathbb{H}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{H}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H^\kappa - q.s. \text{ càdlàg paths, and } \|Y\|_{\mathbb{D}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\sup_{0\leqslant t\leqslant T} |Y_t|^p\right] < +\infty.$$

$\mathbb{I}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes K null at 0 with

$$\mathcal{P}_H^\kappa - q.s. \text{ càdlàg and non-decreasing paths, and } \|K\|_{\mathbb{I}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[(K_T)^p\right] < +\infty.$$

For each $\xi \in L_H^{1,\kappa}$, $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $t\in[0,T]$ denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi] \text{ where } \mathcal{P}_H^\kappa(t^+,\mathbb{P}) := \left\{\mathbb{P}'\in\mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\right\}.$$

Here $\mathbb{E}_t^\mathbb{P}[\xi] := E^\mathbb{P}[\xi|\mathcal{F}_t]$. Then we define for each $p\geqslant\kappa$,

$$\mathbb{L}_H^{p,\kappa} := \left\{\xi\in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < +\infty\right\} \text{ where } \|\xi\|_{\mathbb{L}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^\mathbb{P}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\leqslant t\leqslant T}\left(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right].$$

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi:\Omega\to\mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}}, \text{ for every } 1\leqslant\kappa\leqslant p.$$

4.2.3 Formulation

First, we consider a process S which will play the role of our lower obstacle. We will always assume that S verifies the following properties

(i) S is $\mathbb{F}$-progressively measurable and càdlàg.

(ii) S is uniformly continuous in ω in the sense that for all t

$$|S_t(\omega) - S_t(\widetilde{\omega})| \leqslant \rho\left(\|\omega-\widetilde{\omega}\|_t\right),\ \forall\ (\omega,\widetilde{\omega})\in\Omega^2,$$

for some modulus of continuity ρ and where we define $\|\omega\|_t := \sup_{0\leqslant s\leqslant t}|\omega(s)|$.

Then, we shall consider the following 2RBSDE with lower obstacle S

$$Y_t = \xi - \int_t^T \widehat{F}_s(Y_s,Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0\leqslant t\leqslant T,\ \mathcal{P}_H^\kappa - q.s. \tag{4.2.1}$$

We follow Soner, Touzi and Zhang [101]. For any $\mathbb{P}\in\mathcal{P}_H^\kappa$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi\in\mathbb{L}^2(\mathbb{P})$, let $(y^\mathbb{P},z^\mathbb{P},k^\mathbb{P}) := (y^\mathbb{P}(\tau,\xi), z^\mathbb{P}(\tau,\xi), k^\mathbb{P}(\tau,\xi))$ denote the unique solution to the following standard RBSDE with obstacle S (existence and uniqueness have been proved under our assumptions by Lepeltier and Xu in [68])

$$\begin{cases} y_t^{\mathbb{P}} = \xi - \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s + k_\tau^{\mathbb{P}} - k_t^{\mathbb{P}},\ 0 \leqslant t \leqslant \tau,\ \mathbb{P}-a.s. \\ y_t^{\mathbb{P}} \geqslant S_t,\ \mathbb{P}-a.s. \\ \int_0^T \left(y_{s^-}^{\mathbb{P}} - S_{s^-}\right) dk_s^{\mathbb{P}} = 0,\ \mathbb{P}-a.s. \end{cases} \tag{4.2.2}$$

Definition 4.2.3. *For $\xi \in \mathbb{L}_H^{2,\kappa}$, we say $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to the 2RBSDE (4.2.1) if*

- $Y_T = \xi$, $\mathcal{P}_H^\kappa - q.s.$
- *$\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, the process $K^{\mathbb{P}}$ defined below has nondecreasing paths $\mathbb{P}-a.s.$*

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \leqslant t \leqslant T,\ \mathbb{P}-a.s. \tag{4.2.3}$$

- *We have the following minimum condition*

$$K_t^{\mathbb{P}} - k_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'} - k_T^{\mathbb{P}'}\right],\ 0 \leqslant t \leqslant T,\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{4.2.4}$$

- $Y_t \geqslant S_t$, $\mathcal{P}_H^\kappa - q.s.$

Remark 4.2.3. *In our proof of existence, we will actually show, using recent results of Nutz [86], that the family $\left(K^{\mathbb{P}}\right)_{\mathbb{P} \in \mathcal{P}_H^\kappa}$ can always be aggregated into a universal process K.*

Following [101], in addition to Assumption 4.2.1, we will always assume

Assumption 4.2.2. *The processes $\widehat{F}^0$ and S satisfy the following integrability conditions*

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0 \leqslant t \leqslant T}{\operatorname{ess\,sup}^{\mathbb{P}}} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\int_0^T |\hat{F}_s^0|^\kappa ds\right]\right)^{\frac{2}{\kappa}}\right] < +\infty \tag{4.2.5}$$

$$\psi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0 \leqslant t \leqslant T}{\operatorname{ess\,sup}^{\mathbb{P}}} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\left(\sup_{0 \leqslant s \leqslant T} (S_s)^+\right)^\kappa\right]\right)^{\frac{2}{\kappa}}\right] < +\infty. \tag{4.2.6}$$

4.2.4 Connection with standard RBSDEs

If H is linear in γ, that is to say

$$H_t(y, z, \gamma) := \frac{1}{2}\mathrm{Tr}\left[a_t^0 \gamma\right] - f_t(y, z),$$

where $a^0 : [0, T] \times \Omega \to \mathbb{S}_d^{>0}$ is $\mathbb{F}$-progressively measurable and has uniform upper and lower bounds. As in [101], we no longer need to assume any uniform continuity in ω in this case. Besides, the domain of F is restricted to a^0 and we have

$$\widehat{F}_t(y, z) = f_t(y, z).$$

If we further assume that there exists some $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that $\widehat{a}$ and a^0 coincide $\mathbb{P}-a.s.$ and $\mathbb{E}^{\mathbb{P}}\left[\int_0^T |f_t(0,0)|^2 dt\right] < +\infty$, then $\mathcal{P}_H^\kappa = \{\mathbb{P}\}$.

Then, unlike with 2BSDEs, it is not immediate from the minimum condition (4.2.4) that the process $K^{\mathbb{P}} - k^{\mathbb{P}}$ is actually null. However, we know that $K^{\mathbb{P}} - k^{\mathbb{P}}$ is a martingale with finite variation. Since $\mathbb{P}$ satisfy the martingale representation property, this martingale is also continuous, and therefore it is null. Thus we have

$$0 = k^{\mathbb{P}} - K^{\mathbb{P}},\ \mathbb{P}-a.s.,$$

and the 2RBSDE is equivalent to a standard RBSDE. In particular, we see that the part of $K^{\mathbb{P}}$ which increases only when $Y_{t^-} > S_{t^-}$ is null, which means that $K^{\mathbb{P}}$ satisfies the usual Skorohod condition with respect to the obstacle.

4.3 Uniqueness of the solution and other properties

4.3.1 Representation and uniqueness of the solution

We have similarly as in Theorem 4.4 of [101]

Theorem 4.3.1. *Let Assumptions 4.2.1 and 4.2.2 hold. Assume $\xi \in \mathbb{L}_H^{2,\kappa}$ and that (Y,Z) is a solution to 2RBSDE (4.2.1). Then, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $0 \leqslant t_1 < t_2 \leqslant T$,*

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+,\mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P}-a.s. \tag{4.3.1}$$

Consequently, the 2RBSDE (4.2.1) has at most one solution in $\mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$.

Remark 4.3.1. *Let us now justify the minimum condition (4.2.4). Assume for the sake of clarity that the generator $\widehat{F}$ is equal to 0. By the above Theorem, we know that if there exists a solution to the 2RBSDE (4.2.1), then the process Y has to satisfy the representation (4.3.1). Therefore, we have a natural candidate for a possible solution of the 2RBSDE. Now, assume that we could construct such a process Y satisfying the representation (4.3.1) and which has the decomposition (4.2.1). Then, taking conditional expectations in $Y - y^{\mathbb{P}}$, we end up with exactly the minimum condition (4.2.4).*

Proof. The proof follows the lines of the proof of Theorem 4.4 in [101].

First,

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})} y_t^{\mathbb{P}'}(T,\xi),\ t \in [0,T],\ \mathbb{P}-a.s.,\ \text{for all } \mathbb{P} \in \mathcal{P}_H^\kappa,$$

and thus is unique. Then, since we have that $d\langle Y, B\rangle_t = Z_t d\langle B\rangle_t$, $\mathcal{P}_H^\kappa - q.s.$, Z is unique. Finally, the process $K^{\mathbb{P}}$ is uniquely determined. We shall now prove (4.3.1).

(i) Fix $0 \leqslant t_1 < t_2 \leqslant T$ and $\mathbb{P} \in \mathcal{P}_H^\kappa$. For any $\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$, we have

$$Y_t = Y_{t_2} - \int_t^{t_2} \widehat{F}_s(Y_s, Z_s)ds - \int_t^{t_2} Z_s dB_s + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'},\ t_1 \leqslant t \leqslant t_2,\ \mathbb{P}' - a.s.$$

Now, it is clear that we can always decompose the nondecreasing process $K^{\mathbb{P}}$ into

$$K_t^{\mathbb{P}'} = A_t^{\mathbb{P}'} + B_t^{\mathbb{P}'},\ \mathbb{P}' - a.s.,$$

were $A^{\mathbb{P}'}$ and $B^{\mathbb{P}'}$ are two nondecreasing processes such that $A^{\mathbb{P}'}$ only increases when $Y_{t^-} = S_{t^-}$ and $B^{\mathbb{P}'}$ only increases when $Y_{t^-} > S_{t^-}$. With that decomposition, we can apply a generalization of the usual comparison theorem proved by El Karoui et al. (see Theorem 5.2 in [35]), whose proof is postponed to the Appendix, under $\mathbb{P}'$ to obtain $Y_{t_1} \geqslant y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$ and $A_{t_2}^{\mathbb{P}'} - A_{t_1}^{\mathbb{P}'} \leqslant k_{t_2}^{\mathbb{P}'} - k_{t_1}^{\mathbb{P}'},\ \mathbb{P}' - a.s.$ Since $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t^+$, we get $Y_{t_1} \geqslant y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$ and thus

$$Y_{t_1} \geqslant \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_H^\kappa$. We will show in (iii) below that

$$C_{t_1}^{\mathbb{P}} := \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - k_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'} + k_{t_1}^{\mathbb{P}'}\right)^2\right] < +\infty,\ \mathbb{P} - a.s.$$

For every $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$, denote

$$\delta Y := Y - y^{\mathbb{P}'}(t_2, Y_{t_2}),\ \delta Z := Z - z^{\mathbb{P}'}(t_2, Y_{t_2}) \text{ and } \delta K^{\mathbb{P}'} := K^{\mathbb{P}'} - k^{\mathbb{P}'}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption 4.2.1(iii), there exist two bounded processes λ and η such that for all $t_1 \leqslant t \leqslant T_2$

$$\delta Y_t = \int_t^{t_2} \left(\lambda_s \delta Y_s + \eta_s \widehat{a}_s^{1/2} \delta Z_s\right) ds - \int_t^{t_2} \delta Z_s dB_s + \delta K_{t_2}^{\mathbb{P}'} - \delta K_{t_1}^{\mathbb{P}'},\ \mathbb{P}' - a.s.$$

Define for $t_1 \leqslant t \leqslant t_2$ the following continuous process

$$M_t := \exp\left(\int_{t_1}^t \left(\lambda_s - \frac{1}{2}|\eta_s|^2\right) ds + \int_{t_1}^t \eta_s \widehat{a}_s^{-1/2} dB_s\right),\ \mathbb{P}' - a.s.$$

Note that since λ and η are bounded, we have for all $p \geqslant 1$

$$\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\sup_{t_1 \leqslant t \leqslant t_2} (M_t)^p + \sup_{t_1 \leqslant t \leqslant t_2} (M_t^{-1})^p\right] \leqslant C_p,\ \mathbb{P}' - a.s. \tag{4.3.2}$$

Then, by Itô's formula, we obtain

$$\delta Y_{t_1} = \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\int_{t_1}^{t_2} M_t d\delta K_t^{\mathbb{P}'}\right]. \tag{4.3.3}$$

Let us now prove that the process $K^{\mathbb{P}^{'}} - k^{\mathbb{P}^{'}}$ is nondecreasing. By the minimum condition (4.2.4), it is clear that it is actually a $\mathbb{P}^{'}$-submartingale. Let us apply the Doob-Meyer decomposition under $\mathbb{P}^{'}$, we get the existence of a $\mathbb{P}^{'}$-martingale $N^{\mathbb{P}^{'}}$ and a nondecreasing process $P^{\mathbb{P}^{'}}$, both null at 0, such that

$$K_t^{\mathbb{P}^{'}} - k_t^{\mathbb{P}^{'}} = N_t^{\mathbb{P}^{'}} + P_t^{\mathbb{P}^{'}},\ \mathbb{P}^{'} - a.s.$$

Then, since we know that all the probability measures in $\mathcal{P}_H^\kappa$ satisfy the martingale representation property, the martingale $N^{\mathbb{P}^{'}}$ is continuous. Besides, by the above equation, it also has finite variation. Hence, we have $N^{\mathbb{P}^{'}} = 0$, and the result follows.

Returning back to (4.3.3), we can now write

$$\begin{aligned}
\delta Y_{t_1} &\leqslant \mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\sup_{t_1 \leqslant t \leqslant t_2} (M_t)\left(\delta K_{t_2}^{\mathbb{P}^{'}} - \delta K_{t_1}^{\mathbb{P}^{'}}\right)\right] \\
&\leqslant \left(\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\sup_{t_1 \leqslant t \leqslant t_2} (M_t)^3\right]\right)^{1/3}\left(\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\left(\delta K_{t_2}^{\mathbb{P}^{'}} - \delta K_{t_1}^{\mathbb{P}^{'}}\right)^{3/2}\right]\right)^{2/3} \\
&\leqslant C(C_{t_1}^{\mathbb{P}})^{1/3}\left(\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\delta K_{t_2}^{\mathbb{P}^{'}} - \delta K_{t_1}^{\mathbb{P}^{'}}\right]\right)^{1/3},\ \mathbb{P} - a.s.
\end{aligned}$$

By taking the essential infimum in $\mathbb{P}^{'} \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})$ on both sides and using the minimum condition (4.2.4), we obtain the reverse inequality.

(iii) It remains to show that the estimate for $C_{t_1}^{\mathbb{P}}$ holds. But by definition, we clearly have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^{'}}\left[\left(K_{t_2}^{\mathbb{P}^{'}} - k_{t_2}^{\mathbb{P}^{'}} - K_{t_1}^{\mathbb{P}^{'}} + k_{t_1}^{\mathbb{P}^{'}}\right)^2\right] &\leqslant C\left(\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \phi_H^{2,\kappa}\right) \\
&\quad + C\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leqslant t \leqslant T}\left|y_t^{\mathbb{P}}\right|^2 + \int_0^T\left|\widehat{a}_t^{1/2} z_s^{\mathbb{P}}\right|^2 ds\right] \\
&< +\infty,
\end{aligned}$$

since the last term on the right-hand side is finite thanks to the integrability assumed on ξ and $\widehat{F}^0$.

Then we can proceed exactly as in the proof of Theorem 4.4 in [101]. □

Finally, the following comparison Theorem follows easily from the classical one for RBSDEs (see for instance Theorem 5.2 in [35] and Theorem 3.4 in [68]) and the representation (4.3.1).

Theorem 4.3.2. *Let (Y, Z) and (Y', Z') be the solutions of 2RBSDEs with terminal conditions ξ and $\xi^{'}$, lower obstacles S and $S^{'}$ and generators $\widehat{F}$ and $\widehat{F}^{'}$ respectively (with the corresponding functions H and $H^{'}$), and let $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}})$ and $(y'^{\mathbb{P}}, z'^{\mathbb{P}}, k'^{\mathbb{P}})$ the solutions of the associated RBSDEs. Assume that they both verify our Assumptions 4.2.1 and 4.2.2, that $\mathcal{P}_H^\kappa \subset \mathcal{P}_{H'}^\kappa$ and that we have*

- $\xi \leqslant \xi'$, $\mathcal{P}_H^\kappa - q.s.$
- $\widehat{F}_t(y_t'^{\mathbb{P}}, z_t'^{\mathbb{P}}) \geqslant \widehat{F}_t'(y_t'^{\mathbb{P}}, z_t'^{\mathbb{P}})$, $\mathbb{P} - a.s.$, *for all* $\mathbb{P} \in \mathcal{P}_H^\kappa$.
- $S_t \leqslant S_t'$, $\mathcal{P}_H^\kappa - q.s.$

Then $Y \leqslant Y'$, $\mathcal{P}_H^\kappa - q.s.$

Remark 4.3.2. *Note that in our context, in the above comparison Theorem, even if the obstacles* S *and* S' *are identical, we cannot compare the nondecreasing processes* $K^{\mathbb{P}}$ *and* $K'^{\mathbb{P}}$. *This is due to the fact that the processes* $K^{\mathbb{P}}$ *do not satisfy the Skorohod condition, since it can be considered, at least formally, to come from the addition of a nondecreasing process due to the fact that we work with 2BSDEs, and a nondecreasing process due to the reflection constraint. And only the second one is bound to satisfy the Skorohod condition.*

4.3.2 Some properties of the solution

Now that we have proved the representation (4.3.1), we can show, as in the classical framework, that the solution Y of the 2RBSDE is linked to an optimal stopping problem

Proposition 4.3.1. *Let* (Y, Z) *be the solution to the above* 2*RBSDE* (4.2.1). *Then for each* $t \in [0, T]$ *and for all* $\mathbb{P} \in \mathcal{P}_H^\kappa$

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \ \underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \ \mathbb{E}_t^{\mathbb{P}'}\left[-\int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'})ds + S_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}\right], \ \mathbb{P} - a.s. \quad (4.3.4)$$

$$= \underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \ \mathbb{E}_t^{\mathbb{P}}\left[-\int_t^\tau \widehat{F}_s(Y_s, Z_s)ds + A_\tau^{\mathbb{P}} - A_t^{\mathbb{P}} + S_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}\right], \ \mathbb{P} - a.s. \quad (4.3.5)$$

where $\mathcal{T}_{t,T}$ *is the set of all stopping times valued in* $[t, T]$ *and where* $A_t^{\mathbb{P}} := \int_0^t 1_{\{Y_{s^-} > S_{s^-}\}} dK_s^{\mathbb{P}}$ *is the part of* $K^{\mathbb{P}}$ *which only increases when* $Y_{s^-} > S_{s^-}$.

Remark 4.3.3. *We want to highlight here that unlike with classical RBSDEs, considering an upper obstacle in our context is fundamentally different from considering a lower obstacle. Indeed, having a lower obstacle corresponds, at least formally, to add an nondecreasing process in the definition of a 2BSDE. Since there is already an nondecreasing process in that definition, we still end up with an nondecreasing process. However, in the case of a upper obstacle, we would have to add a non-increasing process in the definition, therefore ending up with a finite variation process. This situation thus becomes much more complicated. Furthermore, in this case we conjecture that the above representation of Proposition 4.3.1 would hold with a sup-inf instead of a sup-sup, indicating that this situation should be closer to stochastic games than to stochastic control. This is an interesting generalization that we leave for future research.*

Proof. By Proposition 3.1 in [68], we know that for all $\mathbb{P} \in \mathcal{P}_H^\kappa$

$$y_t^{\mathbb{P}} = \underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \ \mathbb{E}_t^{\mathbb{P}}\left[-\int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds + S_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}\right], \ \mathbb{P} - a.s.$$

Then the first equality is a simple consequence of the representation formula (4.3.1). For the second one, we proceed exactly as in the proof of Proposition 3.1 in [68]. Fix some $\mathbb{P} \in \mathcal{P}_H^\kappa$ and some $t \in [0,T]$. Let $\tau \in \mathcal{T}_{t,T}$. We obtain by taking conditional expectation in (4.2.1)

$$\begin{aligned} Y_t &= \mathbb{E}_t^{\mathbb{P}}\left[Y_\tau - \int_t^\tau \widehat{F}_s(Y_s,Z_s)ds + K_\tau^{\mathbb{P}} - K_t^{\mathbb{P}}\right] \\ &\geqslant \mathbb{E}_t^{\mathbb{P}}\left[-\int_t^\tau \widehat{F}_s(Y_s,Z_s)ds + S_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} + A_\tau^{\mathbb{P}} - A_t^{\mathbb{P}}\right]. \end{aligned}$$

This implies that

$$Y_t \geqslant \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}}\left[-\int_t^\tau \widehat{F}_s(Y_s,Z_s)ds + A_\tau^{\mathbb{P}} - A_t^{\mathbb{P}} + S_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}\right],\ \mathbb{P}-a.s.$$

Fix some $\varepsilon > 0$ and define the stopping time $D_t^{\mathbb{P},\varepsilon} := \inf\{u \geqslant t,\ Y_u \leqslant S_u + \varepsilon,\ \mathbb{P}-a.s.\} \wedge T$. It is clear by definition that on the set $\left\{D_t^{\mathbb{P},\varepsilon} < T\right\}$, we have $Y_{D_t^{\mathbb{P},\varepsilon}} \leqslant S_{D_t^{\mathbb{P},\varepsilon}} + \varepsilon$. Similarly, on the set $\left\{D_t^{\mathbb{P},\varepsilon} = T\right\}$, we have $Y_s > S_s + \varepsilon$, for all $t \leqslant s \leqslant T$. Hence, for all $s \in [t, D_t^{\mathbb{P},\varepsilon}]$, we have $Y_{s^-} > S_{s^-}$. This implies that $K_{D_t^{\mathbb{P},\varepsilon}} - K_t = A_{D_t^{\mathbb{P},\varepsilon}} - A_t$, and therefore

$$Y_t \leqslant \mathbb{E}_t^{\mathbb{P}}\left[-\int_t^{D_t^{\mathbb{P},\varepsilon}} \widehat{F}_s(Y_s,Z_s)ds + A_{D_t^{\mathbb{P},\varepsilon}}^{\mathbb{P}} - A_t^{\mathbb{P}} + S_{D_t^{\mathbb{P},\varepsilon}} 1_{\left\{D_t^{\mathbb{P},\varepsilon}<T\right\}} + \xi 1_{\left\{D_t^{\mathbb{P},\varepsilon}=T\right\}}\right] + \varepsilon,$$

which ends the proof by arbitrariness of ε. $\square$

We now show that we can obtain more information about the non-decreasing processes $K^{\mathbb{P}}$.

Proposition 4.3.2. *Let Assumptions 4.2.1 and 4.2.2 hold. Assume $\xi \in \mathbb{L}_H^{2,\kappa}$ and $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to the 2RBSDE (4.2.1). Let $\left\{(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}})\right\}_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ be the solutions of the corresponding BSDEs (4.2.2). Then we have the following result. For all $t \in [0,T]$,*

$$\int_0^t 1_{\{Y_{s^-}=S_{s^-}\}} dK_s^{\mathbb{P}} = \int_0^t 1_{\{Y_{s^-}=S_{s^-}\}} dk_s^{\mathbb{P}},\ \mathbb{P}-a.s.$$

Proof. Let us fix a given $\mathbb{P} \in \mathcal{P}_H^\kappa$. Let τ_1 and τ_2 be two $\mathbb{P}$-stopping times such that for all $t \in [\tau_1, \tau_2)$, $Y_{t^-} = S_{t^-}$, $\mathbb{P}-a.s.$

First, by the representation formula (4.3.1), we necessarily have for all $\mathbb{P}$, $Y_{t^-} \geqslant y_{t^-}^{\mathbb{P}}$, $\mathbb{P}-a.s.$ for all t. Moreover, since we also have $y_t^{\mathbb{P}} \geqslant S_t$ by definition, this implies, since all the processes here are càdlàg, that we must have

$$Y_{t^-} = y_{t^-}^{\mathbb{P}} = S_{t^-},\ t \in [\tau_1, \tau_2),\ \mathbb{P}-a.s.$$

Using the fact that Y and $y^{\mathbb{P}}$ solve respectively a 2BSDE and a BSDE, we also have

$$S_{t^-} + \Delta Y_t = Y_t = Y_u - \int_t^u \widehat{F}_s(Y_s,Z_s)ds - \int_t^u Z_s dB_s + K_u^{\mathbb{P}} - K_t^{\mathbb{P}},\ \tau_1 \leqslant t \leqslant u < \tau_2,\ \mathbb{P}-a.s.,$$

and

$$S_{t^-}+\Delta y_t^{\mathbb{P}}=Y_t=y_u^{\mathbb{P}}-\int_t^u\widehat{F}_s(y_s^{\mathbb{P}},z_s^{\mathbb{P}})ds-\int_t^u z_s^{\mathbb{P}}dB_s+k_u^{\mathbb{P}}-k_t^{\mathbb{P}},\ \tau_1\leqslant t\leqslant u<\tau_2,\ \mathbb{P}-a.s.$$

Identifying the martingale parts above, we obtain that $Z_s=z_s^{\mathbb{P}}$, $\mathbb{P}-a.s.$ for all $s\in[t,u]$. Then, identifying the finite variation parts, we have

$$\Delta Y_u-\Delta Y_t-\int_t^u\widehat{F}_s(Y_s,Z_s)ds+K_u^{\mathbb{P}}-K_t^{\mathbb{P}}=\Delta y_u^{\mathbb{P}}-\Delta y_t^{\mathbb{P}}-\int_t^u\widehat{F}_s(y_s^{\mathbb{P}},z_s^{\mathbb{P}})ds+k_u^{\mathbb{P}}-k_t^{\mathbb{P}}.$$

Now, we clearly have

$$\int_t^u\widehat{F}_s(Y_s,Z_s)ds=\int_t^u\widehat{F}_s(y_s^{\mathbb{P}},z_s^{\mathbb{P}})ds,$$

since $Z_s=z_s^{\mathbb{P}}$, $\mathbb{P}-a.s.$ and $Y_{s^-}=y_{s^-}^{\mathbb{P}}=S_{s^-}$ for all $s\in[t,u]$. Moreover, since $Y_{s^-}=y_{s^-}^{\mathbb{P}}=S_{s^-}$ for all $s\in[t,u]$ and since all the processes are càdlàg, the jumps of Y and $y^{\mathbb{P}}$ are equal to the jumps of S. Therefore, we can further identify the finite variation part to obtain

$$K_u^{\mathbb{P}}-K_t^{\mathbb{P}}=k_u^{\mathbb{P}}-k_t^{\mathbb{P}},$$

which is the desired result. □

Remark 4.3.4. *Recall that at least formally, the role of the non-decreasing processes $K^{\mathbb{P}}$ is on the one hand to keep the solution of the 2RBSDE above the obstacle S and on the other hand to keep it above the corresponding RBSDE solutions $y^{\mathbb{P}}$, as confirmed by the representation formula (4.3.1). What the above result tells us is that if Y becomes equal to the obstacle, then it suffices to push it exactly as in the standard RBSDE case. This is conform to the intuition. Indeed, when Y reaches S, then all the $y^{\mathbb{P}}$ are also on the obstacle, therefore, there is no need to counter-balance the second order effects.*

Remark 4.3.5. *The above result leads us naturally to think that one could decompose the non-decreasing process $K^{\mathbb{P}}$ into two non-decreasing processes $A^{\mathbb{P}}$ and $V^{\mathbb{P}}$ such that $A^{\mathbb{P}}$ satisfies the usual Skorohod condition and $V^{\mathbb{P}}$ satisfies*

$$V_t^{\mathbb{P}}=\underset{\mathbb{P}'\in\mathcal{P}_H^{\kappa}(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[V_T^{\mathbb{P}'}\right],\ 0\leqslant t\leqslant T,\ \mathbb{P}-a.s.,\ \forall\mathbb{P}\in\mathcal{P}_H^{\kappa}.$$

Such a decomposition would isolate the effects due to the obstacle and the ones due to the second-order. Of course, the choice $A^{\mathbb{P}}:=k^{\mathbb{P}}$ would be natural, given the minimum condition (4.2.4). However the situation is not that simple. Indeed, we know that

$$\int_0^t 1_{\{Y_{s^-}=S_{s^-}\}}dK_s^{\mathbb{P}}=\int_0^t 1_{\{Y_{s^-}=S_{s^-}\}}dk_s^{\mathbb{P}}.$$

But $k^{\mathbb{P}}$ can increase when Y is strictly above the obstacle, since we can have $Y_{t^-}>y_{t^-}^{\mathbb{P}}=S_{t^-}$. We can thus only write

$$K_t^{\mathbb{P}}=\int_0^t 1_{\{Y_{s^-}=S_{s^-}\}}k_s^{\mathbb{P}}+V_t^{\mathbb{P}}.$$

Then $V^{\mathbb{P}}$ satisfies the minimum condition (4.2.4) when $Y_{t^-}=S_{t^-}$ and when $y_{t^-}^{\mathbb{P}}>S_{t^-}$. However, we cannot say anything when $Y_{t^-}>y_{t^-}^{\mathbb{P}}=S_{t^-}$. The existence of such a decomposition, which is also related to the difficult problem of the Doob-Meyer decomposition for the G-submartingales of Peng [89], is therefore still an open problem.

As a Corollary of the above result, if we have more information on the obstacle S, we can give a more explicit representation for the processes $K^{\mathbb{P}}$. The proof comes directly from the above Proposition and Proposition 4.2 in [37].

Assumption 4.3.1. *S is a semi-martingale of the form*

$$S_t = S_0 + \int_0^t U_s ds + \int_0^t V_s dB_s + C_t,\ \mathcal{P}_H^\kappa - q.s.$$

where C is càdlàg process of integrable variation such that the measure dC_t is singular with respect to the Lebesgue measure dt and which admits the following decomposition

$$C_t = C_t^+ - C_t^-,$$

where C^+ and C^- are nondecreasing processes. Besides, U and V are respectively $\mathbb{R}$ and $\mathbb{R}^d$-valued $\mathcal{F}_t$ progressively measurable processes such that

$$\int_0^T (|U_t| + |V_t|^2)dt + C_T^+ + C_T^- < +\infty,\ \mathcal{P}_H^\kappa - q.s.$$

Corollary 4.3.1. *Let Assumptions 4.2.1, 4.2.2 and 4.3.1 hold. Let (Y, Z) be the solution to the 2RBSDE (4.2.1), then*

$$Z_t = V_t,\ dt \times \mathcal{P}_H^\kappa - q.s. \text{ on the set } \{Y_{t^-} = S_{t^-}\}, \tag{4.3.6}$$

and there exists a progressively measurable process $(\alpha_t^{\mathbb{P}})_{0 \leqslant t \leqslant T}$ such that $0 \leqslant \alpha \leqslant 1$ and

$$1_{\{Y_{t^-}=S_{t^-}\}} dK_t^{\mathbb{P}} = \alpha_t^{\mathbb{P}} 1_{\{Y_{t^-}=S_{t^-}\}} \left(\left[\widehat{F}_t(S_t, V_t) - U_t \right]^+ dt + dC_t^- \right).$$

4.3.3 A priori estimates

We conclude this section by showing some *a priori* estimates which will be useful in the sequel.

Theorem 4.3.3. *Let Assumptions 4.2.1 and 4.2.2 hold. Assume $\xi \in \mathbb{L}_H^{2,\kappa}$ and $(Y, Z, K) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{I}_H^{2,\kappa}$ is a solution to the 2RBSDE (4.2.1). Let $\left\{(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}})\right\}_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ be the solutions of the corresponding BSDEs (4.2.2). Then, there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$ such that*

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[(K_T^{\mathbb{P}})^2 \right] \leqslant C \left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} \right),$$

and

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \left\{ \left\|y^{\mathbb{P}}\right\|^2_{\mathbb{D}^2(\mathbb{P})} + \left\|z^{\mathbb{P}}\right\|^2_{\mathbb{H}^2(\mathbb{P})} + \left\|k^{\mathbb{P}}\right\|^2_{\mathbb{I}^2(\mathbb{P})} \right\} \leqslant C \left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} \right).$$

Proof. By Lemma 2 in [49], we know that there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$, such that for all $\mathbb{P}$

$$\left|y_t^{\mathbb{P}}\right| \leqslant C_\kappa \mathbb{E}_t^{\mathbb{P}} \left[|\xi|^\kappa + \int_t^T \left|\widehat{F}_s^0\right|^\kappa ds + \sup_{t \leqslant s \leqslant T} (S_s^+)^\kappa \right]. \tag{4.3.7}$$

Let us note immediately, that in [49], the result is given with an expectation and not a conditional expectation, and more importantly that the process considered are continuous. However, the generalization is easy for the conditional expectation. As far as the jumps are concerned, their proof only uses Itô's formula for smooth convex functions, for which the jump part can been taken care of easily in the estimates. Then, one can follow exactly their proof to get our result.

This immediately provides the estimate for $y^{\mathbb{P}}$. Now by definition of our norms, we get from (4.3.7) and the representation formula (4.3.1) that

$$\|Y\|^2_{\mathbb{D}^{2,\kappa}_H} \leqslant C_\kappa \left(\|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi^{2,\kappa}_H + \psi^{2,\kappa}_H \right). \tag{4.3.8}$$

Now apply Itô's formula to $|Y|^2$ under each $\mathbb{P} \in \mathcal{P}^\kappa_H$. We get as usual for every $\varepsilon > 0$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] &\leqslant C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \int_0^T |Y_t| \left(\left|\widehat{F}^0_t\right| + |Y_t| + \left|\widehat{a}_t^{1/2} Z_t\right| \right) dt\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_t|\, dK^{\mathbb{P}}_t\right] \\
&\leqslant C\left(\|\xi\|_{\mathbb{L}^{2,\kappa}_H} + \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T} |Y_t|^2 + \left(\int_0^T \left|\widehat{F}^0_t\right| dt\right)^2\right]\right) \\
&\quad + \varepsilon\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt + \left|K^{\mathbb{P}}_T\right|^2\right] + \frac{C^2}{\varepsilon}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T} |Y_t|^2\right].
\end{aligned} \tag{4.3.9}$$

Then by definition of our 2RBSDE, we easily have

$$\mathbb{E}^{\mathbb{P}}\left[\left|K^{\mathbb{P}}_T\right|^2\right] \leqslant C_0\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \sup_{0\leqslant t\leqslant T} |Y_t|^2 + \int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt + \left(\int_0^T \left|\widehat{F}^0_t\right| dt\right)^2\right], \tag{4.3.10}$$

for some constant C_0, independent of ε.

Now set $\varepsilon := (2(1+C_0))^{-1}$ and plug (4.3.10) in (4.3.9). One then gets

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] \leqslant C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \sup_{0\leqslant t\leqslant T} |Y_t|^2 + \left(\int_0^T \left|\widehat{F}^0_t\right| dt\right)^2\right].$$

From this and the estimate for Y, we immediately obtain

$$\|Z\|_{\mathbb{H}^{2,\kappa}_H} \leqslant C\left(\|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi^{2,\kappa}_H + \psi^{2,\kappa}_H \right).$$

Then the estimate for $K^{\mathbb{P}}$ comes from (4.3.10). The estimates for $z^{\mathbb{P}}$ and $k^{\mathbb{P}}$ can be proved similarly. □

Theorem 4.3.4. *Let Assumptions 4.2.1 and 4.2.2 hold. For $i = 1, 2$, let (Y^i, Z^i) be the solutions to the 2RBSDE (4.2.1) with terminal condition ξ^i and lower obstacle S. Then,* *there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$ such that*

$$\left\|Y^1 - Y^2\right\|_{\mathbb{D}_H^{2,\kappa}} \leqslant C \left\|\xi^1 - \xi^2\right\|_{\mathbb{L}_H^{2,\kappa}},$$

and

$$\begin{aligned}&\left\|Z^1 - Z^2\right\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}\left|K_t^{\mathbb{P},1} - K_t^{\mathbb{P},2}\right|^2\right] \\ &\leqslant C \left\|\xi^1 - \xi^2\right\|_{\mathbb{L}_H^{2,\kappa}} \left(\left\|\xi^1\right\|_{\mathbb{L}_H^{2,\kappa}} + \left\|\xi^1\right\|_{\mathbb{L}_H^{2,\kappa}} + (\phi_H^{2,\kappa})^{1/2} + (\psi_H^{2,\kappa})^{1/2}\right).\end{aligned}$$

Proof. As in the previous Proposition, we can follow the proof of Lemma 3 in [49], to obtain that there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$, such that for all $\mathbb{P}$

$$\left|y_t^{\mathbb{P},1} - y_t^{\mathbb{P},2}\right| \leqslant C_\kappa \left(\mathbb{E}_t^{\mathbb{P}}\left[\left|\xi^1 - \xi^2\right|^\kappa\right]\right)^{\frac{1}{\kappa}}. \tag{4.3.11}$$

Now by definition of our norms, we get from (4.3.11) and the representation formula (4.3.1) that

$$\left\|Y^1 - Y^2\right\|^2_{\mathbb{D}_H^{2,\kappa}} \leqslant C_\kappa \left\|\xi^1 - \xi^2\right\|^2_{\mathbb{L}_H^{2,\kappa}}. \tag{4.3.12}$$

Applying Itô's formula to $\left|Y^1 - Y^2\right|^2$, under each $\mathbb{P}\in\mathcal{P}_H^\kappa$, leads to

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2}(Z_t^1 - Z_t^2)\right|^2 dt\right] &\leqslant C\mathbb{E}^{\mathbb{P}}\left[\left|\xi^1 - \xi^2\right|^2\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|Y_t^1 - Y_t^2\right| d(K_t^{\mathbb{P},1} - K_t^{\mathbb{P},2})\right] \\ &\quad + C\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|Y_t^1 - Y_t^2\right|\left(\left|Y_t^1 - Y_t^2\right| + |\widehat{a}_t^{1/2}(Z_t^1 - Z_t^2)|\right) dt\right] \\ &\leqslant C\left(\left\|\xi^1 - \xi^2\right\|^2_{\mathbb{L}_H^{2,\kappa}} + \left\|Y^1 - Y^2\right\|^2_{\mathbb{D}_H^{2,\kappa}}\right) \\ &\quad + \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2}(Z_t^1 - Z_t^2)\right|^2 dt\right] \\ &\quad + C\left\|Y^1 - Y^2\right\|_{\mathbb{D}_H^{2,\kappa}}\left(\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^2 \left(K_T^{\mathbb{P},i}\right)^2\right]\right)^{1/2}\end{aligned}$$

The estimate for $(Z^1 - Z^2)$ is now obvious from the above inequality and the estimates of Proposition 4.3.3.

Finally the estimate for the difference of the nondecreasing processes is obvious by definition. $\square$

4.4 A direct existence argument

We have shown in Theorem 4.3.1 that if a solution exists, it will necessarily verify the representation (4.3.1). This gives us a natural candidate for the solution as a supremum of solutions to standard RBSDEs. However, since those BSDEs are all defined on the support of mutually singular probability measures, it seems difficult to define such a supremum, because of the problems raised by the negligible sets. In order to overcome this, Soner, Touzi and Zhang proposed in [101] a pathwise construction of the solution to a 2BSDE. Let us describe briefly their strategy.

The first step is to define pathwise the solution to a standard BSDE. For simplicity, let us consider first a BSDE with a generator equal to 0. Then, we know that the solution is given by the conditional expectation of the terminal condition. In order to define this solution pathwise, we can use the so-called regular conditional probability distribution (r.p.c.d. for short) of Stroock and Varadhan [104]. In the general case, the idea is similar and consists on defining BSDEs on a shifted canonical space.

Finally, we have to prove measurability and regularity of the candidate solution thus obtained, and the decomposition (4.2.1) is obtained through a non-linear Doob-Meyer decomposition. Our aim in this section is to extend this approach to the reflected case. We refer to Section 2.5 in Chapter 2 for notations.

4.4.1 Existence when ξ is in $\mathrm{UC_b}(\Omega)$

When ξ is in $\mathrm{UC_b}(\Omega)$, we know that there exists a modulus of continuity function ρ for ξ, F and S in ω. Then, for any $0 \leqslant t \leqslant s \leqslant T$, $(y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$ and $\omega,\omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$\left|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})\right| \leqslant \rho\left(\|\omega-\omega'\|_t\right), \ \left|\widehat{F}^{t,\omega}_s(\tilde{\omega},y,z) - \widehat{F}^{t,\omega'}_s(\tilde{\omega},y,z)\right| \leqslant \rho\left(\|\omega-\omega'\|_t\right)$$

$$\left|S^{t,\omega}_s(\tilde{\omega}) - S^{t,\omega'}_s(\tilde{\omega})\right| \leqslant \rho\left(\|\omega-\omega'\|_t\right).$$

We then define for all $\omega \in \Omega$

$$\Lambda(\omega) := \sup_{0 \leqslant s \leqslant t} \Lambda_t(\omega), \tag{4.4.1}$$

where

$$\Lambda_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}^{t,\kappa}_H} \left(\mathbb{E}^{\mathbb{P}}\left[\left|\xi^{t,\omega}\right|^2 + \int_t^T |\widehat{F}^{t,\omega}_s(0,0)|^2 ds + \left(\sup_{t \leqslant s \leqslant T}(S^{t,\omega}_s)^+\right)^2\right]\right)^{1/2}.$$

Now since $\widehat{F}^{t,\omega}$ is also uniformly continuous in ω, we have

$$\Lambda(\omega) < \infty \text{ for some } \omega \in \Omega \text{ iff it holds for all } \omega \in \Omega. \tag{4.4.2}$$

Moreover, when Λ is finite, it is uniformly continuous in ω under the $\mathbb{L}^\infty$-norm and is therefore $\mathcal{F}_T$-measurable.

Now, by Assumption 4.2.2, we have

$$\Lambda_t(\omega) < \infty \text{ for all } (t,\omega) \in [0,T]\times\Omega. \tag{4.4.3}$$

To prove existence, we define the following value process V_t pathwise

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}^{t,\kappa}_H} \mathcal{Y}^{\mathbb{P},t,\omega}_t(T,\xi), \text{ for all } (t,\omega) \in [0,T]\times\Omega, \tag{4.4.4}$$

where, for any $(t_1,\omega)\in[0,T]\times\Omega$, $\mathbb{P}\in\mathcal{P}_H^{t_1,\kappa}$, $t_2\in[t_1,T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta\in\mathbb{L}^2(\mathbb{P})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,\eta):=y_{t_1}^{\mathbb{P},t_1,\omega}$, where $\left(y^{\mathbb{P},t_1,\omega},z^{\mathbb{P},t_1,\omega},k^{\mathbb{P},t_1,\omega}\right)$ is the solution of the following RBSDE with lower obstacle $S^{t_1,\omega}$ on the shifted space Ω^{t_1} under $\mathbb{P}$

$$y_s^{\mathbb{P},t_1,\omega}=\eta^{t_1,\omega}-\int_s^{t_2}\widehat{F}_r^{t_1,\omega}\left(y_r^{\mathbb{P},t_1,\omega},z_r^{\mathbb{P},t_1,\omega}\right)dr-\int_s^{t_2}z_r^{\mathbb{P},t_1,\omega}dB_r^{t_1}+k_{t_2}^{\mathbb{P},t_1,\omega}-k_{t_1}^{\mathbb{P},t_1,\omega}\qquad(4.4.5)$$

$$y_t^{\mathbb{P},t_1,\omega}\geqslant S_t^{t_1,\omega},\ \mathbb{P}-a.s.$$

$$\int_{t_1}^{t_2}\left(y_{s^-}^{\mathbb{P},t_1,\omega}-S_{s^-}^{t_1,\omega}\right)dk_s^{\mathbb{P},t_1,\omega}=0,\ \mathbb{P}-a.s.\qquad(4.4.6)$$

In view of the Blumenthal zero-one law, $\mathcal{Y}_t^{\mathbb{P},t,\omega}(T,\xi)$ is constant for any given (t,ω) and $\mathbb{P}\in\mathcal{P}_H^{t,\kappa}$. Moreover, since $\omega_0=0$ for all $\omega\in\Omega$, it is clear that, for the $y^{\mathbb{P}}$ defined in (4.2.2),

$$\mathcal{Y}^{\mathbb{P},0,\omega}(t,\eta)=y^{\mathbb{P}}(t,\eta)\ \text{for all}\ \omega\in\Omega.$$

Remark 4.4.1. *We could have defined our candidate solution in another way, using BSDEs instead of RBSDEs, but with a random time horizon. This is based on the link with optimal stopping given by (4.3.4). Notice that this approach is similar to the one used by Fabre [40] in her PhD thesis when studying 2BSDEs with the Z part of the solution constrained to stay in a convex set. Using this representation as a supremum of BSDEs for a constrained BSDE is particularly efficient, because in general the non-decreasing process added to the solution has no regularity and we cannot obtain stability results. In our case, the two approaches lead to the same result, in particular because the Skorohod condition for the RBSDE allows us to recover stability, as shown in the Lemma below.*

Lemma 4.4.1. *Let Assumptions 4.2.1 and 4.2.2 hold and consider some ξ in $\mathrm{UC_b}(\Omega)$. Then for all $(t,\omega)\in[0,T]\times\Omega$ we have $|V_t(\omega)|\leqslant C(1+\Lambda_t(\omega))$. Moreover, for all $(t,\omega,\omega')\in[0,T]\times\Omega^2$, $|V_t(\omega)-V_t(\omega')|\leqslant C\rho(\|\omega-\omega'\|_t)$. Consequently, V_t is $\mathcal{F}_t$-measurable for every $t\in[0,T]$.*

Proof. (i) For each $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{P}\in\mathcal{P}_H^{t,\kappa}$, let α be some positive constant which will be fixed later and let $\eta\in(0,1)$. By Itô's formula we have, since $\widehat{F}$ is uniformly Lipschitz and since by (4.4.6) $\int_t^T e^{\alpha s}\left(y_{s^-}^{\mathbb{P},t,\omega}-S_{s^-}^{t,\omega}\right)dk_s^{\mathbb{P},t,\omega}=0$

$$\begin{aligned}
e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}\right|^2&+\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2ds\leqslant e^{\alpha T}\left|\xi^{t,\omega}\right|^2+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|\left|\widehat{F}_s^{t,\omega}(0)\right|ds\\
&+2C\int_t^T\left|y_s^{\mathbb{P},t,\omega}\right|\left(\left|y_s^{\mathbb{P},t,\omega}\right|+\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|\right)ds-2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^t\\
&+2\int_t^T e^{\alpha s}S_{s^-}^{t,\omega}dk_s^{\mathbb{P},t,\omega}-\alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2ds
\end{aligned}$$

$$\begin{aligned}
&\leqslant e^{\alpha T}\left|\xi^{t,\omega}\right|^2+\int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(0)\right|^2ds-2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^t+\eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2ds\\
&+\left(2C+C^2+\frac{C^2}{\eta}-\alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2ds+2\sup_{t\leqslant s\leqslant T}e^{\alpha s}(S_s^{t,\omega})^+(k_T^{\mathbb{P},t,\omega}-k_t^{\mathbb{P},t,\omega}).
\end{aligned}$$

Now choose α such that $\nu := \alpha - 2C - C^2 - \frac{C^2}{\eta} \geqslant 0$. We obtain for all $\varepsilon > 0$

$$\begin{aligned} e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}\right|^2 + (1-\eta)\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds \leqslant e^{\alpha T}\left|\xi^{t,\omega}\right|^2 + \int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(0,0)\right|^2 ds \\ + \frac{1}{\varepsilon}\left(\sup_{t\leqslant s\leqslant T} e^{\alpha s}(S_s^{t,\omega})^+\right)^2 \\ + \varepsilon(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega})^2 \\ - 2\int_t^T e^{\alpha s} y_{s^-}^{\mathbb{P},t,\omega} z_s^{\mathbb{P},t,\omega} dB_s^t. \end{aligned} \tag{4.4.7}$$

Taking expectation in (4.4.7) yields

$$\left|y_t^{\mathbb{P},t,\omega}\right|^2 + (1-\eta)\mathbb{E}^{\mathbb{P}}\left[\int_t^T \left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds\right] \leqslant C\Lambda_t(\omega)^2 + \varepsilon\mathbb{E}^{\mathbb{P}}\left[(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega})^2\right].$$

Now by definition, we also have for some constant C_0 independent of ε

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega})^2\right] &\leqslant C_0\mathbb{E}^{\mathbb{P}}\left[\left|\xi^{t,\omega}\right|^2 + \int_t^T \left|\widehat{F}_s^{t,\omega}(0,0)\right|^2 ds + \int_t^T \left|y_s^{\mathbb{P},t,\omega}\right|^2 ds\right] \\ &\quad + \mathbb{E}^{\mathbb{P}}\left[\int_t^T \left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds\right] \\ &\leqslant C_0\left(\Lambda_t(\omega) + \mathbb{E}^{\mathbb{P}}\left[\int_t^T \left|y_s^{\mathbb{P},t,\omega}\right|^2 ds + \int_t^T \left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds\right]\right). \end{aligned}$$

Choosing η small enough and $\varepsilon = \frac{1}{2C_0}$, Gronwall inequality then implies

$$\left|y_t^{\mathbb{P},t,\omega}\right|^2 \leqslant C(1 + \Lambda_t(\omega)).$$

The result then follows from arbitrariness of $\mathbb{P}$.

(ii) The proof is exactly the same as above, except that one has to use uniform continuity in ω of $\xi^{t,\omega}$, $\widehat{F}^{t,\omega}$ and $S^{t,\omega}$. Indeed, for each $(t,\omega) \in [0,T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}_H^{t,\kappa}$, let α be some positive constant which will be fixed later and let $\eta \in (0,1)$. By Itô's formula we have, since $\widehat{F}$ is uniformly Lipschitz

$$
\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}-y_t^{\mathbb{P},t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2 ds\leqslant e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2\\
&+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left(\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|+\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|\right)ds\\
&+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})\right|ds\\
&+2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d(k_s^{\mathbb{P},t,\omega}-k_s^{\mathbb{P},t,\omega'})-\alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2 ds\\
&-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^t
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})\right|^2 ds\\
&+\left(2C+C^2+\frac{C^2}{\eta}-\alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2 ds\\
&+\eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2 ds\\
&-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^t\\
&+2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d(k_s^{\mathbb{P},t,\omega}-k_s^{\mathbb{P},t,\omega'}).
\end{aligned}
$$

By the Skorohod condition (4.4.6), we also have

$$
\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d(k_s^{\mathbb{P},t,\omega}-k_s^{\mathbb{P},t,\omega'})\leqslant\int_t^T e^{\alpha s}(S_{s^-}^{t,\omega}-S_{s^-}^{t,\omega'})d(k_s^{\mathbb{P},t,\omega}-k_s^{\mathbb{P},t,\omega'}).
$$

Now choose α such that $\nu:=\alpha-2C-C^2-\frac{C^2}{\eta}\geqslant 0$. We obtain for all $\varepsilon>0$

$$
\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}-y_t^{\mathbb{P},t,\omega'}\right|^2+(1-\eta)\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2 ds\\
&\leqslant e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})\right|^2 ds\\
&+\frac{1}{\varepsilon}\left(\sup_{t\leqslant s\leqslant T}e^{\alpha s}(S_s^{t,\omega}-S_s^{t,\omega'})^+\right)^2+\varepsilon(k_T^{\mathbb{P},t,\omega}-k_T^{\mathbb{P},t,\omega'}-k_t^{\mathbb{P},t\omega}+k_t^{\mathbb{P},t,\omega'})^2\\
&-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^t.
\end{aligned}
\tag{4.4.8}
$$

The end of the proof is then similar to the previous step, using the uniform continuity in ω of ξ, F and S. □

Then, we show the same dynamic programming principle as Proposition 4.7 in [102]

Proposition 4.4.1. *Under Assumptions 4.2.1, 4.2.2 and for $\xi \in \mathrm{UC_b}(\Omega)$, we have for all $0 \leqslant t_1 < t_2 \leqslant T$ and for all $\omega \in \Omega$*

$$V_{t_1}(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H^{t_1,\kappa}} \mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2, V_{t_2}^{t_1,\omega}).$$

The proof is almost the same as the proof in [102], but we give it for the convenience of the reader.

Proof. Without loss of generality, we can assume that $t_1 = 0$ and $t_2 = t$. Thus, we have to prove

$$V_0(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H^{\kappa}} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

Denote $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(T,\xi), \mathcal{Z}^{\mathbb{P}}(T,\xi), \mathcal{K}^{\mathbb{P}}(T,\xi))$

(i) For any $\mathbb{P} \in \mathcal{P}_H^{\kappa}$, we know by Lemma 4.3 in [102], that for $\mathbb{P}-a.e.\ \omega \in \Omega$, the r.c.p.d. $\mathbb{P}^{t,\omega} \in \mathcal{P}_H^{t,\kappa}$. Now thanks to the paper of Xu and Qian [93], we know that the solution of reflected BSDEs with Lipschitz generators can be constructed via Picard iteration. Thus, it means that at each step of the iteration, the solution can be formulated as a conditional expectation under $\mathbb{P}$. By the properties of the r.p.c.d., this entails that

$$y_t^{\mathbb{P}}(\omega) = \mathcal{Y}_t^{\mathbb{P}^{t,\omega},t,\omega}(T,\xi), \text{ for } \mathbb{P}-a.e.\ \omega \in \Omega. \tag{4.4.9}$$

Hence, by definition of V_t and the comparison principle for RBSDEs, we get that $y_0^{\mathbb{P}} \leqslant \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$. By arbitrariness of $\mathbb{P}$, this leads to

$$V_0(\omega) \leqslant \sup_{\mathbb{P}\in\mathcal{P}_H^{\kappa}} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

(ii) For the other inequality, we proceed as in [102]. Let $\mathbb{P} \in \mathcal{P}_H^{\kappa}$ and $\varepsilon > 0$. By separability of Ω, there exists a partition $(E_t^i)_{i\geqslant 1} \subset \mathcal{F}_t$ such that $\|\omega - \omega'\|_t \leqslant \varepsilon$ for any i and any $\omega, \omega' \in E_t^i$. Now for each i, fix an $\widehat{\omega}_i \in E_t^i$ and let $\mathbb{P}_t^i$ be an $\varepsilon-$optimizer of $V_t(\widehat{\omega}_i)$.

Now if we define for each $n \geqslant 1$, $\mathbb{P}^n := \mathbb{P}^{n,\varepsilon}$ by

$$\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{n} \mathbb{E}^{\mathbb{P}_t^i}\left[1_E^{t,\omega}\right] 1_{E_t^i}\right] + \mathbb{P}(E \cap \widehat{E}_t^n), \text{ where } \widehat{E}_t^n := \cup_{i>n} E_t^i.$$

Then, by the proof of Proposition 4.7 in [102], we know that $\mathbb{P}^n \in \mathcal{P}_H^{\kappa}$. Besides, by Lemma 4.4.1 and its proof, we know that V and $\mathcal{Y}^{\mathbb{P},t,\omega}$ are uniformly continuous in ω and thus

$$\begin{aligned} V_t(\omega) &\leqslant V_t(\widehat{\omega}_i) + C\rho(\varepsilon) \leqslant \mathcal{Y}_t^{\mathbb{P}_t^i,t,\widehat{\omega}_i}(T,\xi) + \varepsilon + C\rho(\varepsilon) \\ &\leqslant \mathcal{Y}_t^{\mathbb{P}_t^i,t,\omega}(T,\xi) + \varepsilon + C\rho(\varepsilon) = \mathcal{Y}_t^{(\mathbb{P}^n)^{t,\omega},t,\omega}(T,\xi) + \varepsilon + C\rho(\varepsilon). \end{aligned}$$

Then, it follows from (4.4.9) that

$$V_t \leqslant y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon),\ \mathbb{P}^n - a.s. \text{ on } \cup_{i=1}^n E_t^i. \tag{4.4.10}$$

Let now $(y^n, z^n, k^n) := (y^{n,\varepsilon}, z^{n,\varepsilon}, k^{n,\varepsilon})$ be the solution of the following RBSDE with lower obstacle S on $[0,t]$

$$y_s^n = \left[y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon)\right] 1_{\cup_{i=1}^n E_t^i} + V_t 1_{\widehat{E}_t^n} - \int_s^t \widehat{F}_r(y_r^n, z_r^n)dr - \int_s^t z_r^n dB_r + k_t^n - k_s^n,\ \mathbb{P} - a.s. \tag{4.4.11}$$

By the comparison principle for RBSDEs, we know that $\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leqslant y_0^n$. Then since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, the equality (4.4.11) also holds $\mathbb{P} - a.s.$ Using the same arguments and notations as in the proof of Lemma 4.4.1, we obtain

$$\left|y_0^n - y_0^{\mathbb{P}^n}\right|^2 \leqslant C\mathbb{E}^{\mathbb{P}}\left[\varepsilon^2 + \rho(\varepsilon)^2 + \left|V_t - y_t^{\mathbb{P}^n}\right|^2 1_{\widehat{E}_t^n}\right].$$

Then, by Lemma 4.4.1, we have

$$\begin{aligned}
\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leqslant y_0^n \leqslant y_0^{\mathbb{P}^n} + C\left(\varepsilon + \rho(\varepsilon) + \left(\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^2 1_{\widehat{E}_t^n}\right]\right)^{1/2}\right) \\
\leqslant V_0(\omega) + C\left(\varepsilon + \rho(\varepsilon) + \left(\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^2 1_{\widehat{E}_t^n}\right]\right)^{1/2}\right).
\end{aligned}$$

Then it suffices to let n go to $+\infty$ and ε to 0. □

Define now for all (t,ω), the $\mathbb{F}^+$-progressively measurable process

$$V_t^+ := \overline{\lim_{r \in \mathbb{Q}\cap(t,T], r\downarrow t}} V_r.$$

We have the following lemma whose proof is postponed to the Appendix

Lemma 4.4.2. *Under the conditions of the previous Proposition, we have*

$$V_t^+ = \lim_{r \in \mathbb{Q}\cap(t,T], r\downarrow t} V_r,\ \mathcal{P}_H^\kappa - q.s.$$

and thus V^+ is càdlàg $\mathcal{P}_H^\kappa - q.s.$.

Proceeding exactly as in Steps 1 et 2 of the proof of Theorem 4.5 in [102], we can then prove that V^+ is a strong reflected $\widehat{F}$-supermartingale. Then, using the Doob-Meyer decomposition proved in the Appendix in Theorem 4.6.2 for all $\mathbb{P}$, we know that there exists a unique ($\mathbb{P} - a.s.$) process $\overline{Z}^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P})$ and unique nondecreasing càdlàg square integrable processes $A^{\mathbb{P}}$ and $B^{\mathbb{P}}$ such that

- $V_t^+ = V_0^+ + \int_0^t \widehat{F}_s(V_s^+, \overline{Z}_s^{\mathbb{P}})ds + \int_0^t \overline{Z}_s^{\mathbb{P}} dB_s - A_t^{\mathbb{P}} - B_t^{\mathbb{P}},\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$
- $V_t^+ \geqslant S_t,\ \mathbb{P} - a.s.\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$

- $\int_0^T (V_{t^-} - S_{t^-})\, dA_t^{\mathbb{P}}$, $\mathbb{P} - a.s.$, $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$.
- $A^{\mathbb{P}}$ and $B^{\mathbb{P}}$ never act at the same time.

We then define $K^{\mathbb{P}} := A^{\mathbb{P}} + B^{\mathbb{P}}$. By Karandikar [58], since V^+ is a càdlàg semimartingale, we can define a universal process $\overline{Z}$ which aggregates the family $\left\{\overline{Z}^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$.

Recall that V^+ is defined pathwise, and so is the Lebesgue integral $\int_0^t \widehat{F}_s(V_s^+, \overline{Z}_s)ds$. With the recent results of Nutz [86], we know that the stochastic integral $\int_0^t \overline{Z}_s dB_s$ can also be defined pathwise. We can therefore define pathwise

$$K_t := V_0^+ - V_t^+ - \int_0^t \widehat{F}_s(V_s^+, \overline{Z}_s)ds + \int_0^t \overline{Z}_s dB_s,$$

and K is an aggregator for the family $\left(K^{\mathbb{P}}\right)_{\mathbb{P} \in \mathcal{P}_H^\kappa}$, that is to say that it coincides $\mathbb{P} - a.s.$ with $K^{\mathbb{P}}$, for every $\mathbb{P} \in \mathcal{P}_H^\kappa$.

We next prove the representation (4.3.1) for V and V^+, and that, as shown in Proposition 4.11 of [102], we actually have $V = V^+$, $\mathcal{P}_H^\kappa - q.s.$, which shows that in the case of a terminal condition in $UC_b(\Omega)$, the solution of the 2RBSDE is actually $\mathbb{F}$-progressively measurable.

Proposition 4.4.2. *Assume that $\xi \in UC_b(\Omega)$. Under Assumptions 4.2.1 and 4.2.2, we have*

$$V_t = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi) \text{ and } V_t^+ = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi), \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Besides, we also have for all t

$$V_t = V_t^+, \ \mathcal{P}_H^\kappa - q.s.$$

Proof. The proof for the representations is the same as the proof of proposition 4.10 in [102], since we also have a stability result for RBSDEs under our assumptions. For the equality between V and V^+, we also refer to the proof of Proposition 4.11 in [102]. □

Therefore, in the sequel we will use V instead of V^+.

Finally, we have to check that the minimum condition (4.2.4) holds. Fix $\mathbb{P}$ in $\mathcal{P}_H^\kappa$ and $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$. By the Lipschitz property of F, we know that there exists bounded processes λ and η such that

$$\begin{aligned} V_t - y_t^{\mathbb{P}'} &= \int_t^T \lambda_s(V_s - y_s^{\mathbb{P}'})ds - \int_t^T \widehat{a}_s^{1/2}(\overline{Z}_s - z_s^{\mathbb{P}'})(\widehat{a}_s^{-1/2}dB_s - \eta_s ds) \\ &\quad + K_T - K_t - k_T^{\mathbb{P}'} + k_t^{\mathbb{P}'}. \end{aligned} \tag{4.4.12}$$

Then, one can define a probability measure $\mathbb{Q}'$ equivalent to $\mathbb{P}'$ such that

$$V_t - y_t^{\mathbb{P}'} = e^{-\int_0^t \lambda_u du} \mathbb{E}_t^{\mathbb{Q}'} \left[\int_t^T e^{\int_0^s \lambda_u du} d(K_s - k_s^{\mathbb{P}'})\right].$$

Now define the following càdlàg nondecreasing processes

$$\overline{K}_s := \int_0^s e^{\int_0^u \lambda_r dr} dK_u,\ \overline{k}_s^{\mathbb{P}'} := \int_0^s e^{\int_0^u \lambda_r dr} dk_u^{\mathbb{P}'}.$$

By the representation (4.3.1), we deduce that the process $\overline{K} - \overline{k}^{\mathbb{P}'}$ is a $\mathbb{Q}'$-submartingale. Using Doob-Meyer decomposition and the fact that all the probability measures we consider satisfy the martingale representation property, we deduce as in Step (ii) of the proof of Theorem 4.3.1 that this process is actually nondecreasing. Then by definition, this entails that the process $K - k^{\mathbb{P}'}$ is also nondecreasing.

Let us denote

$$P_t^{\mathbb{P}'} := K - k^{\mathbb{P}'}.$$

Returning to (4.4.12) and defining a process M as in Step (ii) of the proof of Theorem 4.3.1, we obtain that

$$V_t - y_t^{\mathbb{P}'} = \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_s dP_s^{\mathbb{P}'}\right] \geqslant \mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leqslant s\leqslant T} M_s\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)\right].$$

Then, we have

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}'}\left[P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right] \\
&= \mathbb{E}_t^{\mathbb{P}'}\left[\left(\inf_{t\leqslant s\leqslant T} M_s\right)^{1/3}\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)\left(\inf_{t\leqslant s\leqslant T} M_s\right)^{-1/3}\right] \\
&\leqslant \left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leqslant s\leqslant T} M_s\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)\right]\mathbb{E}_t^{\mathbb{P}'}\left[\sup_{t\leqslant s\leqslant T} M_s^{-1}\right]\mathbb{E}_t^{\mathbb{P}'}\left[\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)^2\right]\right)^{1/3} \\
&\leqslant C\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)^2\right]\right)^{1/3}\left(V_t - y_t^{\mathbb{P}'}\right)^{1/3}.
\end{aligned}$$

Arguing as in Step (iii) of the proof of Theorem 4.3.1, the above inequality shows that we have

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right] = 0,$$

that is to say that the minimum condition (4.2.4) is satisfied.

4.4.2 Main result

We are now in position to state the main result of this section

Theorem 4.4.1. *Let $\xi \in \mathcal{L}_H^{2,\kappa}$. Under Assumptions 4.2.1 and 4.2.2, there exists a unique solution $(Y,Z,K) \in \mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}\times\mathbb{I}_H^{2,\kappa}$ of the* 2RBSDE *(4.2.1).*

Proof. The proof follow the lines of the proof of Theorem 4.7 in [101]. In general for a terminal condition $\xi \in \mathcal{L}_H^{2,\kappa}$, there exists by definition a sequence $(\xi_n)_{n \geqslant 0} \subset \mathrm{UC_b}(\Omega)$ such that

$$\lim_{n\to+\infty} \|\xi_n - \xi\|_{\mathbb{L}_H^{2,\kappa}} = 0 \text{ and } \sup_{n\geqslant 0} \|\xi_n\|_{\mathbb{L}_H^{2,\kappa}} < +\infty.$$

Let (Y^n, Z^n) be the solution to the 2RBSDE (4.2.1) with terminal condition ξ_n and

$$K_t^n := Y_0^n - Y_t^n + \int_0^t \widehat{F}_s(Y_s^n, Z_s^n)ds + \int_0^t Z_s^n dB_s,\ \mathbb{P}-a.s.$$

By the estimates of Proposition 4.3.4, we have as $n, m \to +\infty$

$$\|Y^n - Y^m\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z^n - Z^m\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T} |K_t^n - K_t^m|\right] \leqslant C_\kappa \|\xi_n - \xi_m\|_{\mathbb{L}_H^{2,\kappa}} \\ \to 0.$$

Extracting a subsequence if necessary, we may assume that

$$\|Y^n - Y^m\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z^n - Z^m\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T} |K_t^n - K_t^m|\right] \leqslant \frac{1}{2^n}. \tag{4.4.13}$$

This implies by Markov inequality that for all $\mathbb{P}$ and all $m \geqslant n \geqslant 0$

$$\mathbb{P}\left[\sup_{0\leqslant t\leqslant T}\left\{|Y_t^n - Y_t^m|^2 + |K_t^n - K_t^m|^2\right\} + \int_0^T |\widehat{a}_t^{1/2}(Z_s^n - Z_s^m)|^2 dt > n^{-1}\right] \leqslant Cn2^{-n}. \tag{4.4.14}$$

Define

$$Y := \overline{\lim_{n\to+\infty}}\, Y^n,\ Z := \overline{\lim_{n\to+\infty}}\, Z^n,\ K := \overline{\lim_{n\to+\infty}}\, K^n,$$

where the $\overline{\lim}$ for Z is taken componentwise. All those processes are clearly $\mathbb{F}^+$-progressively measurable.

By (4.4.14), it follows from Borel-Cantelli Lemma that for all $\mathbb{P}$ we have $\mathbb{P}-a.s.$

$$\lim_{n\to+\infty}\left[\sup_{0\leqslant t\leqslant T}\left\{|Y_t^n - Y_t|^2 + |K_t^n - K_t|^2\right\} + \int_0^T |\widehat{a}_t^{1/2}(Z_s^n - Z_s)|^2 dt\right] = 0.$$

It follows that Y is càdlàg, $\mathcal{P}_H^\kappa - q.s.$, and that K is a càdlàg nondecreasing process, $\mathbb{P}-a.s.$ Furthermore, for all $\mathbb{P}$, sending m to infinity in (4.4.13) and applying Fatou's lemma under $\mathbb{P}$ gives us that $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$.

Finally, we can proceed exactly as in the regular case ($\xi \in \mathrm{UC_b}(\Omega)$) to show that the minimum condition (4.2.4) holds. □

4.5 American contingent claims under volatility uncertainty

First let us recall the link between American contingent claims and RBSDEs in the classical framework (see [37] for more details). Let $\mathcal{M}$ be a standard financial complete market (n risky assets and a bond). It is well known that in some constrained cases the pair wealth-portfolio $(X^{\mathbb{P}}, \pi^{\mathbb{P}})$ satisfies:

$$X_t^{\mathbb{P}} = \xi - \int_t^T b(s, X_s^{\mathbb{P}}, \pi_s^{\mathbb{P}})ds - \int_t^T \pi_s^{\mathbb{P}} \sigma_s dW_s$$

where W is a Brownian motion under the underlying probability measure $\mathbb{P}$, b is convex and Lipschitz with respect to (x, π). In addition we assume that the process $(b(t,0,0))_{t \leqslant T}$ is square-integrable and $(\sigma_t)_{t \leqslant T}$, the volatility matrix of the n risky assets, is invertible and its inverse $(\sigma_t)^{-1}$ is bounded. The classical case corresponds to $b(t,x,\pi) = r_t x + \pi.\sigma_t\theta_t$, where θ_t is the risk premium vector.

When the American contingent claim is exercised at a stopping time $\nu \geqslant t$, the yield is given by

$$\tilde{S}_\nu = S_\nu \mathbf{1}_{[\nu < T]} + \xi_T \mathbf{1}_{[\nu = T]}.$$

Let t be fixed and let $\nu \geqslant t$ be the exercising time of the contingent claim. Then, since the market is complete, there exists a unique pair $(X_s^{\mathbb{P}}(\nu, \tilde{S}_\nu), \pi_s^{\mathbb{P}}(\nu, \tilde{S}_\nu)) = (X_s^{\mathbb{P},\nu}, \pi_s^{\mathbb{P},\nu})$ which replicates $\tilde{S}_\nu$, *i.e.*,

$$dX_s^{\mathbb{P},\nu} = b(s, X_s^{\mathbb{P},\nu}, \pi_s^{\mathbb{P},\nu})dt + \pi_s^{\mathbb{P},\nu}\sigma_s dW_s, \;\; s \leqslant \nu; \;\; X_\nu^{\mathbb{P},\nu} = \tilde{S}_\nu.$$

Therefore the price of the contingent claim is given by:

$$Y_t^{\mathbb{P}} = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}} X_t^{\mathbb{P}}(\nu, \tilde{S}_\nu).$$

Then, the link with RBSDE is given by the following Theorem of [37]

Theorem 4.5.1. *There exist $\pi^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P})$ and a nondecreasing continuous process $k^{\mathbb{P}}$ such that for all $t \in [0,T]$*

$$\begin{cases} Y_t^{\mathbb{P}} = \xi - \int_t^T b(s, Y_s^{\mathbb{P}}, \pi_s^{\mathbb{P}})ds - \int_t^T \pi_s^{\mathbb{P}}\sigma_s dW_s + k_T^{\mathbb{P}} - k_t^{\mathbb{P}} \\ Y_t^{\mathbb{P}} \geqslant S_t \\ \int_0^T (Y_t^{\mathbb{P}} - S_t)dk_t^{\mathbb{P}} = 0. \end{cases}$$

Furthermore, the stopping time $D_t^{\mathbb{P}} = \inf\{s \geqslant t, Y_s^{\mathbb{P}} = S_s\} \wedge T$ is optimal after t.

Let us now go back to our uncertain volatility framework. The pricing of European contingent claims has already been treated in this context by Avellaneda, Lévy and Paras in [2], Denis and Martini in [27] with capacity theory and more recently by Vorbrink in [110] using the G-expectation framework.

We still consider a financial market with a bond and d risky asset $L^1 \dots L^d$, whose dynamics are given by

$$\frac{dL_t^i}{Li_t} = \mu_t^i dt + dB_t^i,\ \mathcal{P}_H^\kappa - q.s.\ \forall i = 1 \dots d.$$

Then for every $\mathbb{P} \in \mathcal{P}_H^\kappa$, the wealth process has the following dynamic

$$X_t^{\mathbb{P}} = \xi - \int_t^T b(s, X_s^{\mathbb{P}}, \pi_s^{\mathbb{P}})ds - \int_t^T \pi_s^{\mathbb{P}} dB_s,\ \mathbb{P} - a.s..$$

In order to be in our 2RBSDE framework, we have to assume that the generator b satisfies Assumptions 4.2.1 and 4.2.2. The main difference is that now b must satisfy stronger integrability conditions and also that it has to be uniformly continuous in ω (when we assume that $\widehat{a}$ in the expression of b is constant). For instance, in the classical case recalled above, it means that r and μ must be uniformly continuous in ω, which is the case if for example they are deterministic. We will also assume that $\xi \in \mathcal{L}_H^{2,\kappa}$. Finally, since S is going to be the obstacle, it has to be uniformly continuous in ω.

Following the intuitions in the papers mentioned above, it is natural in our now incomplete market to consider as a superhedging price for our contingent claim

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} Y_t^{\mathbb{P}'},\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa,$$

where $Y_t^{\mathbb{P}}$ is the price at time t of the contingent claim in the complete market mentioned at the beginning, with underlying probability measure $\mathbb{P}$. Notice immediately that we do not claim that this price is the superreplicating price in our context, in the sense that it would be the smallest one for which there exists a strategy which superreplicates the American contingent claim quasi-surely.

The following Theorem is then a simple consequence of the previous one.

Theorem 4.5.2. *There exist $\pi \in \mathbb{H}_H^{2,\kappa}$ and a universal of nondecreasing càdlàg process K such that for all $t \in [0,T]$ and for all $\mathbb{P} \in \mathcal{P}_H^\kappa$*

$$\begin{cases} Y_t = \xi - \int_t^T b(s, Y_s, \pi_s)ds - \int_t^T \pi_s dB_s + K_T - K_t,\ \mathbb{P} - a.s. \\ Y_t \geqslant S_t,\ \mathbb{P} - a.s. \\ K_t - k_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'} \left[K_T - k_T^{\mathbb{P}'} \right],\ \mathbb{P} - a.s. \end{cases}$$

Furthermore, for all ε, the stopping time $D_t^\varepsilon = \inf\{s \geqslant t, Y_s \leqslant S_s + \varepsilon,\ \mathcal{P}_H^\kappa - q.s.\} \wedge T$ is ε-optimal after t. Besides, for all $\mathbb{P}$, if we consider the stopping times $D_t^{\mathbb{P},\varepsilon} = \inf\left\{s \geqslant t, Y_s^{\mathbb{P}} \leqslant S_s + \varepsilon,\ \mathbb{P} - a.s.\right\} \wedge T$, which are ε-optimal for the American contingent claim under each $\mathbb{P}$, then for all $\mathbb{P}$

$$D_t^\varepsilon \geqslant D_t^{\varepsilon,\mathbb{P}},\ \mathbb{P} - a.s. \tag{4.5.1}$$

Proof. The existence of the processes is a simple consequence of Theorem 4.4.1 and the fact that X is the superhedging price of the contingent claim comes from the representation formula (4.3.1). Then, the ε-optimality of D_t^ε and the inequality (4.5.1) are clear by definition. □

Remark 4.5.1. *The formula (4.5.1) confirms the natural intuition that the smallest optimal time to exercise the American contingent claim when the volatility is uncertain is the supremum, in some sense, of all the optimal stopping times for the classical American contingent claim for each volatility scenario.*

Remark 4.5.2. *As explained in Remark 4.3.5, we cannot find a decomposition that would isolate the effects due to the obstacle and the ones due to the second-order. It is not clear neither for the existence of an optimal stopping time. $D_t = \inf\{s \geqslant t, Y_{s^-} \leqslant S_{s^-},\ \mathcal{P}_H^\kappa - q.s.\} \wedge T$ is not optimal after t. Between t and D_t, $K^{\mathbb{P}}$ is reduced to the part related to the second-order. However this part does not verify the minimum condition because it is possible to have $Y_{t^-} > y_{t^-}^{\mathbb{P}} = S_{t^-}$, thus the process $k^{\mathbb{P}}$ is not identically null.*

4.6 Appendix

4.6.1 Technical proof

Proof. [Proof of Lemma 4.4.2] For each $\mathbb{P}$, let $(\bar{\mathcal{Y}}^{\mathbb{P}}, \bar{\mathcal{Z}}^{\mathbb{P}})$ be the solution of the BSDE with generator $\widehat{F}$ and terminal condition ξ at time T. We define

$$\widetilde{V}^{\mathbb{P}} := V - \bar{\mathcal{Y}}^{\mathbb{P}}.$$

Then, $\widetilde{V}^{\mathbb{P}} \geqslant 0$, $\mathbb{P} - a.s.$

For any $0 \leqslant t_1 < t_2 \leqslant T$, let $(y^{\mathbb{P},t_2}, z^{\mathbb{P},t_2}, k^{\mathbb{P},t_2}) := (\mathcal{Y}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{Z}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{K}^{\mathbb{P}}(t_2, V_{t_2}))$. Since we have for $\mathbb{P} - a.e.\ \omega$, $\mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, V_{t_2})(\omega) = \mathcal{Y}^{\mathbb{P},t_1,\omega}(t_2, V_{t_2}^{t_1,\omega})$, we get from Proposition 4.4.1

$$V_{t_1} \geqslant y_{t_1}^{\mathbb{P},t_2},\ \mathbb{P} - a.s.$$

Denote

$$\widetilde{y}_t^{\mathbb{P},t_2} := y_t^{\mathbb{P},t_2} - \bar{\mathcal{Y}}_t^{\mathbb{P}},\ \ \widetilde{z}_t^{\mathbb{P},t_2} := \widehat{a}_t^{1/2}(z_t^{\mathbb{P},t_2} - \bar{\mathcal{Z}}_t^{\mathbb{P}}).$$

Then $\widetilde{V}_{t_1}^{\mathbb{P}} \geqslant \widetilde{y}_{t_1}^{\mathbb{P},t_2}$ and $(\widetilde{y}^{\mathbb{P},t_2}, \widetilde{z}^{\mathbb{P},t_2})$ satisfies the following RBSDE with lower obstacle $S - \bar{\mathcal{Y}}^{\mathbb{P}}$ on $[0, t_2]$

$$\widetilde{y}_t^{\mathbb{P},t_2} = \widetilde{V}_{t_2}^{\mathbb{P}} - \int_t^{t_2} f_s^{\mathbb{P}}(\widetilde{y}_s^{\mathbb{P},t_2}, \widetilde{z}_s^{\mathbb{P},t_2})ds - \int_t^{t_2} \widetilde{z}_s^{\mathbb{P},t_2} dW_s^{\mathbb{P}} + k_{t_2}^{\mathbb{P},t_2} - k_t^{\mathbb{P},t_2},$$

where

$$f_t^{\mathbb{P}}(\omega, y, z) := \widehat{F}_t(\omega, y + \bar{\mathcal{Y}}_t^{\mathbb{P}}(\omega), \widehat{a}_t^{-1/2}(\omega)z + \bar{\mathcal{Z}}_t^{\mathbb{P}}(\omega)) - \widehat{F}_t(\omega, \bar{\mathcal{Y}}_t^{\mathbb{P}}(\omega), \bar{\mathcal{Z}}_t^{\mathbb{P}}(\omega)).$$

By the definition given in the Appendix, $\widetilde{V}^{\mathbb{P}}$ is a positive weak reflected $f^{\mathbb{P}}$-supermartingale under $\mathbb{P}$. Since $f^{\mathbb{P}}(0,0)=0$, we can apply the downcrossing inequality proved in the Appendix in Theorem 4.6.3 to obtain classically that for $\mathbb{P}-a.e.$ ω, the limit

$$\lim_{r\in\mathbb{Q}\cup(t,T],r\downarrow t}\widetilde{V}_r^{\mathbb{P}}(\omega)$$

exists for all t.

Finally, since $\bar{\mathcal{Y}}^{\mathbb{P}}$ is continuous, we get the result. ◻

4.6.2 Reflected g-expectation

In this section, we extend some of the results of Peng [88] concerning g-supersolution of BSDEs to the case of RBSDEs. Let us note that the majority of the following proofs follows straightforwardly from the original proofs of Peng, with some minor modifications due to the added reflection. However, we still provide most of them since, to the best of our knowledge, they do not appear anywhere else in the literature.

In the following, we fix a probability measure $\mathbb{P}$

4.6.2.1 Definitions and first properties

Let us be given the following objects

- A function $g_s(\omega,y,z)$, $\mathbb{F}$-progressively measurable for fixed y and z, uniformly Lipschitz in (y,z) and such that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(0,0)|^2\,ds\right]<+\infty.$$

- A terminal condition ξ which is $\mathcal{F}_T$-measurable and in $L^2(\mathbb{P})$.
- A càdlàg process V with $\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}|V_t|^2\right]<+\infty.$
- A càdlàg process S such that $\mathbb{E}^{\mathbb{P}}\left[\left(\sup_{0\leqslant t\leqslant T}(S_t)^+\right)^2\right]<+\infty.$

We want to study the following problem. Finding $(y,z,k)\in\mathbb{D}^2(\mathbb{P})\times\mathbb{H}^2(\mathbb{P})\times\mathbb{I}^2(\mathbb{P})$ such that

$$\begin{cases} y_t=\xi+\displaystyle\int_t^T g_s(y_s,z_s)ds-\int_t^T z_sdW_s+k_T-k_t+V_T-V_t,\ 0\leqslant t\leqslant T,\ \mathbb{P}-a.s. \\ y_t\geqslant S_t,\ \mathbb{P}-a.s. \\ \displaystyle\int_0^T (y_{s^-}-S_{s^-})\,dk_s=0,\ \mathbb{P}-a.s. \end{cases} \tag{4.6.1}$$

We first have a result of existence and uniqueness

Proposition 4.6.1. *Under the above hypotheses, there exists a unique solution $(y, z, k) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ to the reflected BSDE (4.6.1).*

Proof. Consider the following penalized BSDE, whose existence and uniqueness are ensured by the results of Peng [88]

$$y_t^n = \xi + \int_t^T g_s(y_s^n, z_s^n)ds - \int_t^T z_s^n dW_s + k_T^n - k_t^n + V_T - V_t,$$

where $k_t^n := n \int_0^t (y_s^n - S_s)^- ds$.

Then, define $\widetilde{y}_t^n := y_t^n + V_t$, $\widetilde{\xi} := \xi + V_T$, $\widetilde{z}_t^n := z_t^n$, $\widetilde{k}_t^n := k_t^n$ and $\widetilde{g}_t(y, z) := g_t(y - V, z)$. We have

$$\widetilde{y}_t^n = \widetilde{\xi} - \int_t^T \widetilde{g}_s(\widetilde{y}_s^n, \widetilde{z}_s^n)ds - \int_t^T \widetilde{z}_s^n dW_s + \widetilde{k}_T^n - \widetilde{k}_t^n,$$

Then, since we know by Lepeltier and Xu [68], that the above penalization procedure converges to a solution of the corresponding RBSDE, existence and uniqueness are then simple generalization of the classical results in RBSDE theory. $\square$

We also have a comparison theorem in this context

Proposition 4.6.2. *Let ξ_1 and $\xi_2 \in L^2(\mathbb{P})$, V^i, $i = 1, 2$ be two adapted, càdlàg processes and $g_s^i(\omega, y, z)$ two functions, which all verify the above assumptions. Let $(y^i, z^i, k^i) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$, $i = 1, 2$ be the solutions of the following RBSDEs with lower obstacle S^i*

$$y_t^i = \xi^i + \int_t^T g_s^i(y_s^i, z_s^i)ds - \int_t^T z_s^i dW_s + k_T^i - k_t^i + V_T^i - V_t^i,\ \mathbb{P} - a.s.,\ i = 1, 2,$$

respectively. If

- $\xi_1 \geqslant \xi_2$, $\mathbb{P} - a.s.$
- $V^1 - V^2$ *is nondecreasing,* $\mathbb{P} - a.s.$
- $S^1 \geqslant S^2$, $\mathbb{P} - a.s.$
- $g_s^1(y_s^1, z_s^1) \geqslant g_s^2(y_s^1, z_s^1)$, $dt \times d\mathbb{P} - a.s.$

then it holds $\mathbb{P} - a.s.$ that for all $t \in [0, T]$

$$y_t^1 \geqslant y_t^2.$$

Besides, if $S^1 = S^2$, then we also have $dk^1 \leqslant dk^2$.

Proof. The first part can be proved exactly as in [34], whereas the second one comes from the fact that the penalization procedure converges in this framework, as seen previously. $\square$

Remark 4.6.1. *If we replace the deterministic time T by a stopping time τ, then all the above is still valid.*

From now on, we will specialize the discussion to the case where the process V is actually in $\mathbb{I}^2(\mathbb{P})$ and consider the following RBSDE

$$\begin{cases} y_t = \xi + \int_{t\wedge\tau}^{\tau} g_s(y_s, z_s)ds + V_\tau - V_{t\wedge\tau} + k_\tau - k_{t\wedge\tau} - \int_{t\wedge\tau}^{\tau} z_s dW_s,\ 0 \leqslant t \leqslant \tau,\ \mathbb{P}-a.s. \\ y_t \geqslant S_t,\ \mathbb{P}-a.s. \\ \int_0^{\tau} (y_{s^-} - S_{s^-})\, dk_s = 0,\ \mathbb{P}-a.s. \end{cases} \tag{4.6.2}$$

Definition 4.6.1. *If y is a solution of a RBSDE of the form (4.6.2), then we call y a reflected g-supersolution on $[0,\tau]$. If $V = 0$ on $[0,\tau]$, then we call y a reflected g-solution.*

We now face a first difference from the case of non-reflected supersolution. Since in our case we have two nondecreasing processes, if a g-supersolution is given, there can exist several nondecreasing processes V and k such that (4.6.2) is satisfied. Indeed, we have the following proposition

Proposition 4.6.3. *Given y a g-supersolution on $[0,\tau]$, there is a unique $z \in \mathbb{H}^2(\mathbb{P})$ and a unique couple $(k,V) \in (\mathbb{I}^2(\mathbb{P}))^2$ (in the sense that the sum $k+V$ is unique), such that (y,z,k,V) satisfy (4.6.2). Besides, there exists a unique quadruple (y,z,k',V') satisfying (4.6.2) such that k' and V' never act at the same time.*

Proof. If both (y,z,k,V) and (y,z^1,k^1,V^1) satisfy (4.6.2), then applying Itô's formula to $(y_t - y_t)^2$ gives immediately that $z = z^1$ and thus $k + V = k^1 + V^1$, $\mathbb{P}-a.s.$

Then, if (y,z,k,V) satisfying (4.6.2) is given, then it is easy to construct (k',V') such that

- k' only increases when $y_{t^-} = S_{t^-}$.
- V' only increases when $y_{t^-} > S_{t^-}$.
- $V'_t + k'_t = V_t + k_t$, $dt \times d\mathbb{P} - a.s.$

and such a couple is unique. □

Remark 4.6.2. *We give a counter-example to the general uniqueness in the above Proposition. Let $T = 2$ and consider the following RBSDE*

$$\begin{cases} y_t = -2 + 2 - t + k_2 - k_t - \int_t^2 z_s dW_s,\ 0 \leqslant t \leqslant 2,\ \mathbb{P}-a.s. \\ y_t \geqslant -\frac{t^2}{2},\ \mathbb{P}-a.s. \\ \int_0^2 \left(y_{s^-} + \frac{s^2}{2}\right) dk_s = 0,\ \mathbb{P}-a.s. \end{cases}$$

We then have $z=0$, $y_t = 1_{0\leqslant t\leqslant 1}\left(\frac{1}{2}-t\right) - \frac{t^2}{2}1_{1<t\leqslant 2}$, $V=0$ *and* $k_t = 1_{t\geqslant 1}\frac{t^2-2t+1}{2}$.

However, we can also take

$$y'_t = 1_{t\geqslant 1}\frac{t^2-2t+1}{4} \text{ and } k'_t = 1_{t\geqslant 1}1_{t\geqslant 1}\frac{t^2-2t+1}{4}.$$

Following Peng [88], this allows us to define

Definition 4.6.2. *Let y be a supersolution on $[0,\tau]$ and let (y,z,k,V) be the related unique triple in the sense of the RBSDE (4.6.2), where k and V never act at the same time. Then we call (z,k,V) the decomposition of y.*

4.6.2.2 Monotonic limit theorem

We now study a limit theorem for reflected g-supersolutions, which is very similar to theorems 2.1 and 2.4 of [88].

We consider a sequence of reflected g-supersolutions

$$\begin{cases} y_t^n = \xi^n + \int_t^T g_s(y_s^n, z_s^n)ds + V_T^n - V_t^n + k_T^n - k_t^n - \int_t^T z_s^n dW_s,\ 0\leqslant t\leqslant T,\ \mathbb{P}-a.s. \\ y_t^n \geqslant S_t,\ \mathbb{P}-a.s. \\ \int_0^T \left(y_{s^-}^n - S_{s^-}\right) dk_s^n = 0,\ \mathbb{P}-a.s. \end{cases}$$

where the V^n are in addition supposed to be continuous.

Theorem 4.6.1. *If we assume that (y_t^n) increasingly converges to (y_t) with*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}|y_t|^2\right] < +\infty,$$

and that (k_t^n) decreasingly converges to (k_t), then y is a g-supersolution, that is to say that there exists $(z,V)\in\mathbb{H}^2(\mathbb{P})\times\mathbb{I}^2(\mathbb{P})$ such that

$$\begin{cases} y_t = \xi + \int_t^T g_s(y_s, z_s)ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s,\ 0\leqslant t\leqslant T,\ \mathbb{P}-a.s. \\ y_t \geqslant S_t,\ \mathbb{P}-a.s. \\ \int_0^T (y_{s^-} - S_{s^-})\, dk_s = 0,\ \mathbb{P}-a.s. \end{cases}$$

Besides, z is the weak (resp. strong) limit of z^n in $\mathbb{H}^2(\mathbb{P})$ (resp. in $\mathbb{H}^p(\mathbb{P})$ for $p<2$) and V_t is the weak limit of V_t^n in $L^2(\mathbb{P})$.

Before proving the Theorem, we will need the following Lemma

Lemma 4.6.1. *Under the hypotheses of Theorem 4.6.1, there exists a constant $C>0$ independent of n such that*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |z_s^n|^2\, ds + (V_T^n)^2 + (k_T^n)^2\right] \leqslant C.$$

Proof. We have

$$\begin{aligned}V_T^n + k_T^n &= y_0^n - y_T^n - \int_0^T g_s(y_s^n, z_s^n)ds + \int_0^T z_s^n dW_s \\ &\leqslant C\left(\sup_{0\leqslant t\leqslant T}|y_t^n| + \int_0^T |z_s^n|\, ds + \int_0^T |g_s(0,0)|\, ds + \left|\int_0^T z_s^n dW_s\right|\right). \end{aligned} \tag{4.6.3}$$

Besides, we also have for all $n \geqslant 1$, $y_t^1 \leqslant y_t^n \leqslant y_t$ and thus $|y_t^n| \leqslant |y_t^1| + |y_t|$, which in turn implies that

$$\sup_n \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}|y_t^n|^2\right] \leqslant C.$$

Reporting this in (4.6.3) and using BDG inequality, we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[(V_T^n)^2 + (k_T^n)^2\right] &\leqslant \mathbb{E}^{\mathbb{P}}\left[(V_T^n + k_T^n)^2\right] \\ &\leqslant C_0\left(1 + \mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(0,0)|^2\, ds + \int_0^T |z_s^n|^2\, ds\right]\right). \end{aligned} \tag{4.6.4}$$

Then, using Itô's formula, we obtain classically for all $\varepsilon > 0$

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\int_0^T |z_s^n|^2\, ds\right] &\leqslant \mathbb{E}^{\mathbb{P}}\left[(y_T^n)^2 + 2\int_0^T y_s^n g_s(y_s^n, z_s^n)ds + 2\int_0^T y_{s^-}^n d(V_s^n + k_s^n)\right] \\ &\leqslant \mathbb{E}^{\mathbb{P}}\left[C\left(1 + \sup_{0\leqslant t\leqslant T}|y_t^n|^2\right) + \int_0^T \frac{|z_s^n|^2}{2}ds + \varepsilon\left(|V_T^n|^2 + |k_T^n|^2\right)\right]. \end{aligned} \tag{4.6.5}$$

Then, from (4.6.4) and (4.6.5), we obtain by choosing $\varepsilon = \frac{1}{4C_0}$ that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |z_s^n|^2\, ds\right] \leqslant C.$$

Reporting this in (4.6.4) ends the proof. □

Proof. [Proof of Theorem 4.6.1] By Lemma 4.6.1 and its proof we first have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(y_s^n, z_s^n)|^2\, ds\right] \leqslant C\mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(0,0)|^2 + |y_s^n|^2 + |z_s^n|^2\, ds\right] \leqslant C.$$

Thus $g_s(y_s^n, z_s^n)$ and z^n are bounded in $\mathbb{H}^2(\mathbb{P})$, and there exists subsequences which converge respectively to some g_s and z_s. Therefore, for every stopping time τ, we also have the following weak convergences

$$\int_0^\tau z_s^n dW_s \to \int_0^\tau z_s dW_s,\ \int_0^\tau g_s(y_s^n, z_s^n)ds \to \int_0^\tau \bar{g}_s ds,$$

$$V_\tau^n \to -y_\tau + y_0 - k_\tau - \int_0^\tau \bar{g}_s ds + \int_0^\tau z_s dW_s.$$

Then by the section theorem, it is clear that V and k are nondecreasing, and by Lemma 2.2 of [88] we know that y, V and k are càdlàg. We now show the strong convergence of z^n. Following Peng [88], we apply Itô's formula between two stopping times τ and σ. Since V^n is continuous, we obtain

$$\mathbb{E}^{\mathbb{P}}\left[\int_\sigma^\tau |z_s^n - z_s|^2\, ds\right] \leqslant \mathbb{E}^{\mathbb{P}}\left[|y_\tau^n - y_\tau|^2 + \sum_{\sigma \leqslant t \leqslant \tau} (\Delta(V_t + k_t))^2\right]$$
$$+ 2\mathbb{E}^{\mathbb{P}}\left[\int_\sigma^\tau |y_s^n - y_s|\, |g_s(y_s^n, z_s^n) - \bar{g}_s|\, ds + \int_\sigma^\tau (y_s^n - y_s) d(V_s + k_s)\right].$$

Then we can finish exactly as in [88] to obtain the desired convergence. Since g is supposed to be Lipschitz, we actually have

$$\bar{g}_s = g_s(y_s, z_s),\ \mathbb{P} - a.s.$$

Finally, since for each n, we have $y_t^n \geqslant S_t$, we have $y_t \geqslant S_t$. For the Skorohod condition, we have, since the k^n are decreasing

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T (y_{t^-} - S_{t^-})\, dk_t\right] \leqslant \mathbb{E}^{\mathbb{P}}\left[\int_0^T (y_{t^-} - y_{t^-}^n)\, dk_t + \int_0^T (y_{t^-}^n - S_{t^-})\, dk_t^n\right]$$
$$= \mathbb{E}^{\mathbb{P}}\left[\int_0^T (y_{t^-} - y_{t^-}^n)\, dk_t\right].$$

Then, we have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T (y_{t^-} - y_{t^-}^n)\, dk_t\right] \leqslant \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leqslant t \leqslant T} \left|y_t^1 - y_t\right|^2\right]\right)^{1/2} \left(\mathbb{E}^{\mathbb{P}}\left[k_T^2\right]\right)^{1/2} < +\infty$$

Therefore by Lebesgue dominated convergence Theorem, we obtain that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T (y_{t^-} - y_{t^-}^n)\, dk_t\right] \to 0,$$

and thus

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T (y_{t^-} - S_{t^-})\, dk_t\right] \leqslant 0,$$

which ends the proof. □

4.6.2.3 Doob-Meyer decomposition

We now introduce the notion of reflected g-(super)martingales.

Definition 4.6.3. (i) *A reflected g-martingale on $[0,T]$ is a reflected g-solution on $[0,T]$.*

(ii) *(Y_t) is a reflected g-supermartingale in the strong (resp. weak) sense if for all stopping time $\tau \leqslant T$ (resp. all $t \leqslant T$), we have $\mathbb{E}^{\mathbb{P}}[|Y_\tau|^2] < +\infty$ (resp. $\mathbb{E}^{\mathbb{P}}[|Y_t|^2] < +\infty$) and if the reflected g-solution (y_s) on $[0,\tau]$ (resp. $[0,t]$) with terminal condition Y_τ (resp. Y_t) verifies $y_\sigma \leqslant Y_\sigma$ for every stopping time $\sigma \leqslant \tau$ (resp. $y_s \leqslant Y_s$ for every $s \leqslant t$).*

As in the case without reflection, under mild conditions, a reflected g-supermartingale in the weak sense corresponds to a reflected g-supermartingale in the strong sense. Besides, thanks to the comparison Theorem, it is clear that a g-supersolution on $[0,T]$ is also a g-supermartingale in the weak and strong sense on $[0,T]$. The following Theorem addresses the converse property, which gives us a nonlinear Doob-Meyer decomposition.

Theorem 4.6.2. *Let (Y_t) be a right-continuous reflected g-supermartingale on $[0,T]$ in the strong sense with*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}|Y_t|^2\right] < +\infty.$$

Then (Y_t) is a reflected g-supersolution on $[0,T]$, that is to say that there exists a unique triple $(z,k,V)\in\mathbb{H}^2(\mathbb{P})\times\mathbb{I}^2(\mathbb{P})\times\mathbb{I}^2(\mathbb{P})$ such that

$$\begin{cases} Y_t = Y_T + \int_t^T g_s(Y_s,z_s)ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s,\ 0\leqslant t\leqslant T,\ \mathbb{P}-a.s. \\ Y_t \geqslant S_t,\ \mathbb{P}-a.s. \\ \int_0^T (Y_{s^-} - S_{s^-})\, dk_s = 0,\ \mathbb{P}-a.s. \\ V \text{ and } k \text{ never act at the same time.} \end{cases}$$

We follow again [88] and consider the following sequence of RBSDEs

$$\begin{cases} y_t^n = Y_T + \int_t^T g_s(y_s^n,z_s^n)ds + n\int_t^T (Y_s - y_s^n)ds + k_T^n - k_t^n - \int_t^T z_s^n dW_s,\ 0\leqslant t\leqslant T \\ y_t^n \geqslant S_t,\ \mathbb{P}-a.s. \\ \int_0^T \left(y_{s^-}^n - S_{s^-}\right) dk_s^n = 0,\ \mathbb{P}-a.s. \end{cases}$$

We then have

Lemma 4.6.2. *For all n, we have*

$$Y_t \geqslant y_t^n.$$

Proof. The proof is exactly the same as the proof of Lemma 3.4 in [88], so we omit it.
□

Proof. [Proof of Theorem 4.6.2] The uniqueness is due to the uniqueness for reflected g-supersolutions proved in Proposition 4.6.3. For the existence part, we first notice that since $Y_t \geqslant y_t^n$ for all n, by the comparison Theorem for RBSDEs, we have $y_t^n \leqslant y_t^{n+1}$ and $dk_t^n \geqslant dk_t^{n+1}$. Therefore they converge monotonically to some processes y and k. Besides, y is bounded from above by Y. Therefore, all the conditions of Theorem 4.6.1 are satisfied and y is a reflected g-supersolution on $[0,T]$ of the form

$$y_t = Y_T + \int_t^T g_s(y_s,z_s)ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s,$$

where V_t is the weak limit of $V_t^n := n\int_0^t (Y_s - y_s^n)ds$.

From Lemma 4.6.1, we have

$$\mathbb{E}^{\mathbb{P}}[(V_T^n)^2] = n^2\mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_s - y_s^n|^2\, ds\right] \leqslant C.$$

It then follows that $Y_t = y_t$, which ends the proof. □

4.6.2.4 Downcrossing inequality

In this section we prove a downcrossing inequality for reflected g-supermartingales in the spirit of the one proved in [21]. We use the same notations as in the classical theory of g-martingales (see [21] and [88] for instance).

Theorem 4.6.3. *Assume that $g(0,0) = 0$. Let (Y_t) be a positive reflected g-supermartingale in the weak sense and let $0 = t_0 < t_1 < ... < t_i = T$ be a subdivision of $[0,T]$. Let $0 < a < b$, then there exists $C > 0$ such that $D_a^b[Y,n]$, the number of downcrossings of $[a,b]$ by $\{Y_{t_j}\}$, verifies*

$$\mathcal{E}^{-\mu}[D_a^b[Y,n]] \leqslant \frac{C}{b-a}\mathcal{E}^{\mu}[Y_0 \wedge b],$$

where μ is the Lipschitz constant of g.

Proof. Consider

$$\begin{cases} y_t^i = Y_{t_i} - \int_t^{t_i}(\mu\,|y_s^i| + \mu\,|z_s^i|)ds + k_{t_i}^i - k_t^i - \int_t^{t_i} z_s^i dW_s,\ 0 \leqslant t \leqslant t_i,\ \mathbb{P}-a.s. \\ y_t^i \geqslant S_t,\ \mathbb{P}-a.s. \\ \int_0^{t_i}\left(y_{s^-}^i - S_{s^-}\right)dk_s^i = 0,\ \mathbb{P}-a.s. \end{cases}$$

We define $a_s^i := -\mu \mathrm{sgn}(z_s^i)1_{t_{j-1}<s\leqslant t_j}$ and $a_s := \sum_{i=0}^n a_s^i$. Let $\mathbb{Q}^a$ be the probability measure defined by

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T a_s dW_s\right).$$

We then have easily that $y_t^i \geqslant 0$ since $Y_{t_i} \geqslant 0$ and

$$y_t^i = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,t_i}} \mathbb{E}_t^{\mathbb{Q}^a}\left[e^{-\mu(\tau-t)}S_\tau 1_{\tau<t_i} + Y_{t_i}e^{-\mu(t_i-t)}1_{\tau=t_i}\right].$$

Since Y is reflected g-supermartingale (and thus also a reflected $g^{-\mu}$-supermartingale where $g_s^{-\mu}(y,z) := -\mu(|y|+|z|)$), we therefore obtain

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t_{i-1},t_i}} \mathbb{E}_{t_{i-1}}^{\mathbb{Q}^a}\left[e^{-\mu(\tau-t_{i-1})}S_\tau 1_{\tau<t_i} + Y_{t_i}e^{-\mu(t_i-t_{i-1})}1_{\tau=t_i}\right] \leqslant Y_{t_{i-1}}.$$

Hence, by choosing $\tau = t_j$ above, we get

$$\mathbb{E}_{t_{i-1}}^{\mathbb{Q}^a}\left[Y_{t_i}e^{-\mu(t_i-t_{i-1})}\right] \leqslant Y_{t_{i-1}},$$

which implies that $(e^{-\mu t_i}Y_{t_i})_{0\leqslant i\leqslant n}$ is a $\mathbb{Q}^a$-supermartingale. Then we can finish the proof exactly as in [21]. $\square$

Chapitre 5

Second Order BSDEs With Jumps

5.1 Introduction

In this chapter, we study a class of 2BSDEs with jumps. The rest of the chapter is organized as follows. In Section 5.2, we introduce the set of probability measures on the Skorohod space $\mathbb{D}$ that we will work with. Using the notion of martingale problems on $\mathbb{D}$, we construct probability measures under which the canonical process has given characteristics. Then we prove an aggregation result under this family. Finally, we define the notion of 2BSDEJs and show how it is linked with classic BSDEs with jumps. Section 5.3 is devoted to a uniqueness result and some *a priori* estimates, and Section 5.4 concerns our existence result. In Section 5.5, as an application of previous results, we study a robust exponential utility maximization problem. The Appendix 5.6 is dedicated to the proof of some important technical results. This chapter is based on [60] and [61].

5.2 Preliminaries

Let $\Omega := \mathbb{D}([0,T],\mathbb{R}^d)$ be the space of càdlàg paths defined on $[0,T]$ with values in R^d and such that $w(0) = 0$, equipped with the Skorohod topology, so that it is a complete, separable metric space (see [10] for instance). The uniform norm on Ω is defined by $\|\omega\|_\infty := \sup_{0\leqslant t\leqslant T}|\omega_t|$. We denote B the canonical process, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ the filtration generated by B, $\mathbb{F}^+ := \left\{\mathcal{F}_t^+\right\}_{0\leqslant t\leqslant T}$ the right limit of $\mathbb{F}$ and for any $\mathbb{P}$, $\mathcal{F}_t^\mathbb{P} := \mathcal{F}_t^+ \vee \mathcal{N}^\mathbb{P}(\mathcal{F}_t^+)$ where

$$\mathcal{N}^\mathbb{P}(\mathcal{G}) := \left\{E \in \Omega, \text{ there exists } \widetilde{E} \in \mathcal{G} \text{ such that } E \subset \widetilde{E} \text{ and } \mathbb{P}(\widetilde{E}) = 0\right\}.$$

As usual, for any filtration $\mathbb{G}$ and any probability measure $\mathbb{P}$, $\overline{\mathcal{G}}^\mathbb{P}$ will denote the corresponding completed filtration.

We then define as in [101] a local martingale measure $\mathbb{P}$ as a probability measure such that B is a $\mathbb{P}$-local martingale. Since we are working in the Skorohod space, we can then define the continuous martingale part of B, noted B^c, and its purely discontinuous part, noted B^d, both being local martingales under each local martingale measures (see [56]). We then associate to the jumps of B a counting measure μ_{B^d}, which is a random measure on $\mathcal{B}(\mathbb{R}^+)\times E$ (where $E := \mathbb{R}^r\backslash\{0\}$ for some $r\in\mathbb{N}^*$), defined pathwise by

$$\mu_{B^d}([0,t],A) := \sum_{0<s\leqslant t} \mathbf{1}_{\{\Delta B_s^d\in A\}},\ \forall t\geqslant 0,\ \forall A\subset E. \tag{5.2.1}$$

We also denote by $\nu_s^{\mathbb{P}}(ds,dx)$ the compensator of $\mu_{B^d}(ds,dx)$, which is a predictable random measure, under $\mathbb{P}$ and by $\widetilde{\mu}^{\mathbb{P}}_{B^d}(ds,dx)$ the corresponding compensated measure.

We then denote $\overline{\mathcal{P}}_W$ the set of all local martingale measures $\mathbb{P}$ such that $\mathbb{P}$-a.s.

(i) The quadratic variation of B^c is absolutely continuous with respect to the Lebesgue measure dt and its density takes values in $\mathbb{S}_d^{>0}$.

(ii) The compensator $\nu_t^{\mathbb{P}}(dt,dx)$ under $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure dt.

In this discontinuous setting, we will say that a probability measure $\mathbb{P}\in\overline{\mathcal{P}}_W$ satisfies the martingale representation property if for any $(\overline{\mathbb{F}}^{\mathbb{P}},\mathbb{P})$-local martingale M, there exists a unique $\overline{\mathbb{F}}^{\mathbb{P}}$-predictable processes H and a unique $\overline{\mathbb{F}}^{\mathbb{P}}$-predictable function U such that $(H,U)\in\mathbb{H}^2_{loc}(\mathbb{P})\times\mathbb{J}^2_{loc}(\mathbb{P})$ (those spaces are defined later) and

$$M_t = M_0 + \int_0^t H_s dB_s^c + \int_0^t \int_E U_s(x)\widetilde{\mu}^{\mathbb{P}}_{B^d}(ds,dx).$$

We now follow [103] and introduce their so-called universal filtration. For this we let $\mathcal{P}$ be a given subset of $\overline{\mathcal{P}}_W$, we define

Definition 5.2.1. (i) *A property is said to hold $\mathcal{P}$-quasi-surely ($\mathcal{P}$-q.s. for short), if it holds $\mathbb{P}-a.s.$ for all $\mathbb{P}\in\mathcal{P}$.*

(ii) *We call $\mathcal{P}$-polar sets the elements of $\mathcal{N}_{\mathcal{P}} := \cap_{\mathbb{P}\in\mathcal{P}}\mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$.*

Then, we define as in [103]

$$\widehat{\mathbb{F}}^{\mathcal{P}} := \left\{\widehat{\mathcal{F}}_t^{\mathcal{P}}\right\}_{t\geqslant 0} \text{ where } \widehat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}}\left(\mathcal{F}_t^{\mathbb{P}}\vee\mathcal{N}_{\mathcal{P}}\right).$$

Finally, we let $\mathcal{T}$ and $\widehat{\mathcal{T}}^{\mathcal{P}}$ the sets of all $\mathbb{F}$ and $\widehat{\mathbb{F}}^{\mathcal{P}}$ stopping times, and we recall that thanks to Lemma 2.4 in [103] we do not have to worry about the universal filtration not being complete under each $\mathbb{P}\in\mathcal{P}$.

5.2.1 Issues related to aggregation

5.2.1.1 The main problem

A crucial issue in the definition of the 2BSDEs in [101] is the aggregation of the quadratic variation of the canonical process B under a wide family of probability measures.

Let $\mathcal{P}\subset\overline{\mathcal{P}}_W$ be a set of non necessarily dominated probability measures and let $\{X^{\mathbb{P}},\ \mathbb{P}\in\mathcal{P}\}$ be a family of random variables indexed by $\mathcal{P}$. One can think for example of the stochastic integrals $X_t^{\mathbb{P}} := {}^{(\mathbb{P})}\int_0^t H_s dB_s$, where $\{H_t,\ t\geqslant 0\}$ is a predictable process.

Definition 5.2.2. *An aggregator of the family* $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ *is a random variable* $\hat{X}$ *such that*

$$\hat{X} = X^{\mathbb{P}}, \ \mathbb{P} - a.s, \text{ for every } \mathbb{P} \in \mathcal{P}.$$

Bichteler [9], Karandikar [58], or more recently Nutz [86] all showed in different contexts, and under different assumptions, that it is possible to find an aggregator for the Itô stochastic integrals ${}^{(\mathbb{P})}\!\int_0^t H_s dB_s$.

A direct consequence of this result is the possibility to aggregate the quadratic variation process $\{[B,B]_t,\, t \geqslant 0\}$. Indeed, using Itô's formula, we can write

$$[B,B]_t = B_t B_t^T - 2\int_0^t B_{s^-} dB_s^T$$

and the aggregation of the stochastic integrals automatically yields the aggregation of the bracket $\{[B,B]_t,\, t \geqslant 0\}$.

This also allows us to give a pathwise definition of the process $\widehat{a}$, which is an aggregator for the density of the quadratic variation of the continuous part of B, by

$$\widehat{a}_t := \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left(\langle B^c \rangle_t - \langle B^c \rangle_{t-\varepsilon} \right),$$

Soner, Touzi and Zhang, motivated by the study of stochastic target problems under volatility uncertainty, obtained in [103] an aggregation result for a family of probability measures corresponding to the laws of some continuous martingales on the canonical space $\Omega = \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$, under a *separability* assumption on the quadratic variations (see their definition 4.8) and under an additional *consistency* condition (which is usually only necessary) for the family to aggregate.

To define correctly the notion of 2BSDEJs, we need to aggregate not only the quadratic variation $[B,B]$ of the canonical process, but also its compensated jump measure. However, this predictable compensator is usually obtained thanks to the Doob-Meyer decomposition of the submartingale $[B,B]$. It is therefore clear that this compensator depends explicitly on the underlying probability measure, and it is not clear at all whether an aggregator always exists or not. This is a first main difference with the continuous case. In order to solve this problem, we follow the spirit of [103] and restrict our set of probability measures (by adding an analogous separability condition for jump measures) so as to generalize some of their results of [103] to the case of processes with jumps.

After these first notations, in the following subsection, in order to construct a probability measure under which the canonical process has a given quadratic variation and a given jump measure, we will use the notion of martingale problem for semimartingales with general characteristics, as defined in the book by Jacod and Shiryaev [56] to which we refer.

5.2.1.2 Characterization by martingale problems

In this subsection, we extend the connection between diffusion processes and probability measures established in [103] thanks to weak solutions of SDEs, to our general jump case with the more general notion of martingale problems.

Let $\mathcal{N}$ be the set of $\mathbb{F}$-predictable random measures ν on $\mathcal{B}(E)$ satisfying

$$\int_0^T \int_E (1 \wedge |x|^2)\nu_s(dx)ds < +\infty \text{ and } \int_0^T \int_{|x|>1} |x|\, \nu_s(dx)ds < +\infty, \ \forall \omega \in \Omega, \tag{5.2.2}$$

and let $\mathcal{D}$ be the set of $\mathbb{F}$-predictable processes α taking values in $\mathbb{S}_d^{>0}$ with

$$\int_0^T |\alpha_t| dt < +\infty, \text{ for every } \omega \in \Omega.$$

We define a martingale problem as follows

Definition 5.2.3. *For $\mathbb{F}$-stopping times τ_1 and τ_2, for $(\alpha, \nu) \in \mathcal{D}\times\mathcal{N}$ and for a probability measure $\mathbb{P}_1$ on $\mathcal{F}_{\tau_1}$, we say that $\mathbb{P}$ is a solution of the martingale problem $(\mathbb{P}_1, \tau_1, \tau_2, \alpha, \nu)$ if*

(i) $\mathbb{P} = \mathbb{P}_1$ *on* $\mathcal{F}_{\tau_1}$.

(ii) *The canonical process B on $[\tau_1, \tau_2]$ is a semimartingale under $\mathbb{P}$ with characteristics*

$$\left(-\int_{\tau_1}^{\cdot} \int_E x \mathbf{1}_{|x|>1} \nu_s(dx)ds, \int_{\tau_1}^{\cdot} \alpha_s ds, \nu_s(dx)ds \right).$$

Remark 5.2.1. *We refer to Theorem III.2.7 in [56] for the fact that $\mathbb{P}$ is a solution of the martingale problem $(\mathbb{P}_1, \tau_1, \tau_2, \alpha, \nu)$ if and only if the following properties hold:*

(i) $\mathbb{P} = \mathbb{P}_1$ *on* $\mathcal{F}_{\tau_1}$.

(ii) *The processes M, J and L defined below are $\mathbb{P}$-local martingales on $[\tau_1, \tau_2]$*

$$M_t := B_t - \sum_{\tau_1 \leqslant s \leqslant t} \mathbf{1}_{|\Delta B_s|>1} \Delta B_s + \int_{\tau_1}^t x \mathbf{1}_{|x|>1} \nu_s(dx)ds, \ \tau_1 \leqslant t \leqslant \tau_2$$
$$J_t := M_t^2 - \int_{\tau_1}^t \alpha_s ds - \int_{\tau_1}^t \int_{|x| \leqslant 1} x^2 \nu_s(dx)ds, \ \tau_1 \leqslant t \leqslant \tau_2$$
$$Q_t := \int_{\tau_1}^t \int_E g(x)\mu_B(ds, dx) - \int_{\tau_1}^t \int_E g(x)\nu_s(dx)ds, \ \tau_1 \leqslant t \leqslant \tau_2, \ \forall g \in \mathcal{C}^+(\mathbb{R}^d).$$

We say that the martingale problem associated to (α, ν) has a unique solution if, for every stopping times τ_1, τ_2 and for every probability measure $\mathbb{P}_1$, the martingale problem $(\mathbb{P}_1, \tau_1, \tau_2, \alpha, \nu)$ has a unique solution.

Let now $\overline{\mathcal{A}}_W$ be the set of $(\alpha, \nu) \in \mathcal{D} \times \mathcal{N}$, such that there exists a solution to the martingale problem $(\mathbb{P}_1, 0, +\infty, \alpha, \nu)$, where $\mathbb{P}_1$ is such that $\mathbb{P}_1(B_0 = 0) = 1$.

We also denote by $\mathcal{A}_W$ the set of $(\alpha,\nu)\in\overline{\mathcal{A}}_W$ such that there exists a unique solution to the martingale problem $(\mathbb{P}_1,0,+\infty,\alpha,\nu)$, where $\mathbb{P}_1$ is such that $\mathbb{P}_1(B_0=0)=1$. Denote $\mathbb{P}^\alpha_\nu$ this unique solution. Finally we set

$$\mathcal{P}_W := \{\mathbb{P}^\alpha_\nu,\ (\alpha,\nu)\in\mathcal{A}_W\}.$$

Remark 5.2.2. *We take here as an initial condition that $B_0=0$. This does not generate a loss of generality, since at the end of the day, the probability measures under which we are going to work will all satisfy the Blumenthal $0-1$ law. Hence, B_0 will have to be a constant and we choose 0 for simplicity.*

5.2.1.3 Notations and definitions

Following [103], for $a,b\in\mathcal{D}$ and $\nu_1,\nu_2\in\mathcal{N}$, we define the first disagreement times as follows

$$\begin{aligned}
\theta^{a,b} &:= \inf\left\{t\geqslant 0,\ \int_0^t a_s ds \neq \int_0^t b_s ds\right\},\\
\theta_{\nu_1,\nu_2} &:= \inf\left\{t\geqslant 0,\ \int_0^t\int_E x\nu^1_s(dx)ds \neq \int_0^t\int_E x\nu^2_s(dx)ds\right\}\\
\theta^{a,b}_{\nu_1,\nu_2} &:= \theta^{a,b}\wedge\theta_{\nu_1,\nu_2}.
\end{aligned}$$

For every $\widehat{\tau}$ in $\widehat{\mathcal{F}}^{\mathcal{P}}$, we define the following event

$$\Omega^{a,\nu_1,b,\nu_2}_{\widehat{\tau}} := \left\{\widehat{\tau}<\theta^{a,b}_{\nu_1,\nu_2}\right\}\cup\left\{\widehat{\tau}=\theta^{a,b}_{\nu_1,\nu_2}=+\infty\right\}.$$

Finally, we introduce the following notion inspired by [103]

Definition 5.2.4. *$\mathcal{A}_0\subset\mathcal{A}_W$ is a generating class of coefficients if*

(i) *$\mathcal{A}_0$ is stable for the concatenation operation, i.e. if $(a,\nu_1),(b,\nu_2)\in\mathcal{A}_0\times\mathcal{A}_0$ then for each t,*

$$\left(a\mathbf{1}_{[0,t]}+b\mathbf{1}_{[t,+\infty)},\nu_1\mathbf{1}_{[0,t]}+\nu_2\mathbf{1}_{[t,+\infty)}\right)\in\mathcal{A}_0.$$

(ii) *For every $(a,\nu_1),(b,\nu_2)\in\mathcal{A}_0\times\mathcal{A}_0$, $\theta^{a,b}_{\nu_1,\nu_2}$ is a constant. Or equivalently, for each t, $\Omega^{a,\nu_1,b,\nu_2}_t$ equals Ω or $\emptyset$.*

Definition 5.2.5. *We say that $\mathcal{A}$ is a separable class of coefficients generated by $\mathcal{A}_0$ if $\mathcal{A}_0$ is a generating class of coefficients and if $\mathcal{A}$ consists of all processes (a,ν) of the form*

$$a=\sum_{n=0}^{+\infty}\sum_{i=1}^{+\infty}a^n_i\mathbf{1}_{E^n_i}\mathbf{1}_{[\tau_n,\tau_{n+1})}\ \text{and}\ \nu=\sum_{n=0}^{+\infty}\sum_{i=1}^{+\infty}\nu^n_i\mathbf{1}_{\tilde{E}^n_i}\mathbf{1}_{[\tilde{\tau}_n,\tilde{\tau}_{n+1})},\tag{5.2.3}$$

where for each i and for each n, $(a^n_i,\nu^n_i)\subset\mathcal{A}_0$, τ_n and $\tilde{\tau}_n$ are $\mathcal{F}$-stopping times with $\tau_0=0$, such that

(i) *$\tau_n<\tau_{n+1}$ on $\{\tau_n<+\infty\}$.*

(ii) $\inf\{n \geqslant 0,\, \tau_n = +\infty\} < \infty$.

(iii) τ_n *takes countably many values in some fixed* $I_0 \subset [0,T]$ *which is countable and dense in* $[0,T]$.

(iv) *For each* n, $(E_i^n)_{i \geqslant 1} \subset \mathcal{F}_{\tau_n}$ *and* $(\tilde{E}_i^n)_{i \geqslant 1} \subset \mathcal{F}_{\tilde{\tau}_n}$ *form a partition of* Ω.

Remark 5.2.3. *If we refine the subdivisions, we can always take a common sequence of stopping times* $(\tau_n)_{n \geqslant 0}$ *and common sets* $(E_i^n)_{i \geqslant 1, n \geqslant 0}$ *for* a *and for* ν. *Moreover, the definition indeed depends on the countable subset* I_0 *introduced above. We acknowledge that as in [103] this set could be changed, but for the sake of clarity, it will be fixed throughout the chapter. We will also show in Section 5.4.4 that this has only limited impact on our results. For practical purposes, one could take for instance* $I_0 = \mathbb{Q} \cap [0,T]$.

Example 5.2.1. $\widetilde{\mathcal{A}}_0$ *composed of deterministic processes* a *and* ν *forms a generating class of coefficients.*

The following Proposition generalizes Proposition 4.11 of [103] and shows that a separable class of coefficients inherits the "good" properties of its generating class.

Proposition 5.2.1. *Let* $\mathcal{A}$ *be a separable class of coefficients generated by* $\mathcal{A}_0$. *Then*

(i) *If* $\mathcal{A}_0 \subset \mathcal{A}_W$, *then* $\mathcal{A} \subset \mathcal{A}_W$.

(ii) $\mathcal{A}$*-quasi surely is equivalent to* $\mathcal{A}_0$*-quasi surely.*

(iii) *If every* $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha,\, (\alpha,\nu) \in \mathcal{A}_0\}$ *satisfies the martingale representation property, then every* $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha,\, (\alpha,\nu) \in \mathcal{A}\}$ *also satisfies the martingale representation property.*

(iv) *If every* $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha,\, (\alpha,\nu) \in \mathcal{A}_0\}$ *satisfies the Blumenthal* $0-1$ *law, then every* $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha,\, (\alpha,\nu) \in \mathcal{A}\}$ *also satisfies the Blumenthal* $0-1$ *law.*

As in [103], to prove this result, we need two Lemmas. The first one is a straightforward generalization of Lemma 4.12 in [103], so we omit the proof. The second one is analogous to Lemma 4.13 in [103].

Lemma 5.2.1. *Let* $\mathcal{A}$ *be a separable class of coefficients generated by* $\mathcal{A}_0$. *For any* $(a,\nu) \in \mathcal{A}$, *and any* $\mathcal{F}$*-stopping time* $\tau \in \mathcal{T}$, *there exist* $\tilde{\tau} \in \mathcal{T}$ *with* $\tilde{\tau} \geqslant \tau$, *a sequence* $(a_i,\nu_i)_{i \geqslant 1} \subset \mathcal{A}_0$ *and a partition* $(E_i)_{i \geqslant 1} \subset \mathcal{F}_\tau$ *of* Ω *such that* $\tilde{\tau} > \tau$ *on* $\{\tau < +\infty\}$ *and*

$$a_t = \sum_{i \geqslant 1} a_i(t) \mathbf{1}_{E_i} \text{ and } \nu_t = \sum_{i \geqslant 1} \nu_i(t) \mathbf{1}_{E_i},\ t < \tilde{\tau}. \tag{5.2.4}$$

In particular, $E_i \subset \Omega_{\tilde{\tau}}^{a,\nu,a_i,\nu_i}$ *which implies that* $\cup_n \Omega_{\tilde{\tau}}^{a,\nu,a_i,\nu_i} = \Omega$. *Finally, if* a *and* ν *take the form (5.2.3) and* $\tau \geqslant \tau_n$, *then we can choose* $\tilde{\tau} \geqslant \tau_{n+1}$.

Proof. We refer to the proof of lemma 4.12 in [103]. □

Lemma 5.2.2. *Let $\tau_1, \tau_2 \in \mathcal{T}$ be two stopping times such that $\tau_1 \leqslant \tau_2$, and $(a_i, \nu_i)_{i \geqslant 1} \subset \overline{\mathcal{A}}_W$ and let $\{E_i,\, i \geqslant 1\} \subset \mathcal{F}_{\tau_1}$ be a partition of Ω. Finally let $\mathbb{P}^0$ be a probability measure on $\mathcal{F}_{\tau_1}$ and let $\{\mathbb{P}^i,\, i \geqslant 1\}$ be a sequence of probability measures such that for each i, $\mathbb{P}^i$ is a solution of the martingale problem $(\mathbb{P}^0, \tau_1, \tau_2, a_i, \nu_i)$. Define*

$$\mathbb{P}(E) := \sum_{i \geqslant 1} \mathbb{P}^i(E \cap E_i) \text{ for all } E \in \mathcal{F}_{\tau_2},$$

$$a_t := \sum_{i \geqslant 1} a_i(t)\mathbf{1}_{E_i} \text{ and } \nu_t := \sum_{i \geqslant 1} \nu_i(t)\mathbf{1}_{E_i},\ t \in [\tau_1, \tau_2].$$

Then $\mathbb{P}$ is a solution of the martingale problem $(\mathbb{P}^0, \tau_1, \tau_2, a, \nu)$.

Proof. By definition, $\mathbb{P} = \mathbb{P}^0$ on $\mathcal{F}_{\tau_1}$. In view of remark 5.2.1, it is enough to prove that M, J and Q are $\mathbb{P}$-local martingales on $[\tau_1, \tau_2]$. By localizing if necessary, we may assume as usual that all these processes are actually bounded. For any stopping times $\tau_1 \leqslant R \leqslant S \leqslant \tau_2$, and any bounded $\mathcal{F}_R$-measurable random variable η, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left([M_S - M_R]\eta\right) &= \sum_{i \geqslant 1} \mathbb{E}^{\mathbb{P}^i}\left([M_S - M_R]\eta\mathbf{1}_{E_i}\right) \\
&= \sum_{i \geqslant 1} \mathbb{E}^{\mathbb{P}^i}\left(\mathbb{E}^{\mathbb{P}^i}([M_S - M_R]|\mathcal{F}_R)\eta\mathbf{1}_{E_i}\right) = 0.
\end{aligned}$$

Thus M is a $\mathbb{P}$-local martingale on $[\tau_1, \tau_2]$. We can prove in exactly the same manner that J and Q are also $\mathbb{P}$-local martingales on $[\tau_1, \tau_2]$ and the proof is complete. □

Proof. [Proof of Proposition 5.2.1] The proof follows closely the proof of Proposition 4.11 in [103] and we give it for the convenience of the reader.

(i) We take $(a, \nu) \in \mathcal{A}$, let us prove that $(a, \nu) \in \mathcal{A}_W$.

We fix two stopping times θ_1, θ_2 in $\mathcal{T}$ and a probability measure $\mathbb{P}^0$ on $\mathcal{F}_{\theta_1}$. We define a sequence $(\tilde{\tau}_n)_{n \geqslant 0}$ as follows:

$$\tilde{\tau}_0 := \theta_1 \text{ and } \tilde{\tau}_n := (\tau_n \vee \theta_1) \wedge \theta_2,\ n \geqslant 1.$$

To prove that the martingale problem $(\mathbb{P}^0, \theta_1, \theta_2, a, \nu)$ has a unique solution, we prove by induction on n that the martingale problem $(\mathbb{P}^0, \tilde{\tau}_0, \tilde{\tau}_n, a, \nu)$ has a unique solution.

Step 1 of the induction: Let $n = 1$, and let us first construct a solution to the martingale problem $(\mathbb{P}^0, \tilde{\tau}_0, \tilde{\tau}_1, a, \nu)$. For this purpose, we apply Lemma 5.2.1 with $\tau = \tilde{\tau}_0$ and $\tilde{\tau} = \tilde{\tau}_1$, which leads to $a_t = \sum_{i \geqslant 1} a_i(t)\mathbf{1}_{E_i}$ and $\nu_t = \sum_{i \geqslant 1} \nu_i(t)\mathbf{1}_{E_i}$ for all $t < \tilde{\tau}_1$, where $(a_i, \nu_i) \in \mathcal{A}_0$ and $\{E_i,\, i \geqslant 1\} \subset \mathcal{F}_{\tilde{\tau}_0}$ form a partition of Ω. For $i \geqslant 1$, let $\mathbb{P}^{0,i}$ be the unique solution of the martingale problem $(\mathbb{P}^0, \tilde{\tau}_0, \tilde{\tau}_1, a_i, \nu_i)$ and define

$$\mathbb{P}^{0,a}(E) := \sum_{i \geqslant 1} \mathbb{P}^{0,i}(E \cap E_i) \text{ for all } E \in \mathcal{F}_{\tilde{\tau}_1}.$$

Lemma 5.2.2 tells us that $\mathbb{P}^{0,a}$ solves the martingale problem $(\mathbb{P}^0, \tilde\tau_0, \tilde\tau_1, a, \nu)$. Now let $\mathbb{P}$ be an arbitrary solution of the martingale problem $(\mathbb{P}^0, \tilde\tau_0, \tilde\tau_1, a, \nu)$, and let us prove that $\mathbb{P} = \mathbb{P}^{0,a}$. We first define

$$\mathbb{P}^i(E) := \mathbb{P}(E \cap E_i) + \mathbb{P}^{0,i}(E \cap E_i^c),\ \forall E \in \mathcal{F}_{\tilde\tau_1}.$$

Using Lemma 5.2.2, and the facts that $a_i = a\mathbf{1}_{E_i} + a_i\mathbf{1}_{E_i^c}$ and $\nu_i = \nu\mathbf{1}_{E_i} + \nu_i\mathbf{1}_{E_i^c}$, we conclude that $\mathbb{P}^i$ solves the martingale problem $(\mathbb{P}^0, \tilde\tau_0, \tilde\tau_1, a_i, \nu_i)$. This problem having a unique solution, we have $\mathbb{P}^i = \mathbb{P}^{0,i}$ on $\mathcal{F}_{\tilde\tau_1}$. This implies that for each $i \geqslant 1$ and for each $E \in \mathcal{F}_{\tilde\tau_1}$, $\mathbb{P}^i(E \cap E_i) = \mathbb{P}^{0,i}(E \cap E_i)$, and finally

$$\mathbb{P}^{0,a}(E) = \sum_{i \geqslant 1} \mathbb{P}^{0,i}(E \cap E_i) = \sum_{i \geqslant 1} \mathbb{P}^i(E \cap E_i) = \mathbb{P}(E),\ \forall E \in \mathcal{F}_{\tilde\tau_1}.$$

Step 2 of the induction: We assume that the martingale problem $(\mathbb{P}^0, \tilde\tau_0, \tilde\tau_n, a, \nu)$ has a unique solution denoted $\mathbb{P}^n$. Using the same reasoning as above, we see that the martingale problem $(\mathbb{P}^n, \tilde\tau_n, \tilde\tau_{n+1}, a, \nu)$ has a unique solution, denoted $\mathbb{P}^{n+1}$. Then the processes M, J and Q defined in Remark 5.2.1 are $\mathbb{P}^{n+1}$-local martingales on $[\tilde\tau_n, \tilde\tau_{n+1}]$, and since $\mathbb{P}^{n+1}$ coincides with $\mathbb{P}^n$ on $\mathcal{F}_{\tilde\tau_n}$, M, J and Q are also $\mathbb{P}^{n+1}$-local martingales on $[\tilde\tau_0, \tilde\tau_n]$. And hence $\mathbb{P}^{n+1}$ solves the martingale problem $(\mathbb{P}^0, \tilde\tau_0, \tilde\tau_{n+1}, a, \nu)$. We suppose now that $\mathbb{P}$ is another arbitrary solution to the problem $(\mathbb{P}^0, \tilde\tau_0, \tilde\tau_{n+1}, a, \nu)$. By the induction assumption, $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_{\tilde\tau_n}$, then $\mathbb{P}$ solves the problem $(\mathbb{P}^n, \tilde\tau_n, \tilde\tau_{n+1}, a, \nu)$, and by uniqueness $\mathbb{P} = \mathbb{P}^{n+1}$ on $\mathcal{F}_{\tilde\tau_{n+1}}$. The induction is now complete.

Remark that $\mathcal{F}_{\theta_2} = \vee_{n \geqslant 1}\mathcal{F}_{\tilde\tau_n}$. Indeed, since $\inf\{n \geqslant 1\ :\ \tau_n = +\infty\} < +\infty$, then $\inf\{n \geqslant 1\ :\ \tilde\tau_n = \theta_2\} < +\infty$. This allows to define $\mathbb{P}^\infty(E) := \mathbb{P}^n(E)$ for $E \in \mathcal{F}_{\tilde\tau_n}$ and to extend it uniquely to $\mathcal{F}_{\theta_2}$. Now using again Remark 5.2.1, we conclude that $\mathbb{P}^\infty$ solves $(\mathbb{P}^0, \theta_1, \theta_2, a, \nu)$ and is unique.

(ii) We now prove that $\mathcal{A}$-quasi surely is equivalent to $\mathcal{A}_0$-quasi surely.

We take $(a, \nu) \in \mathcal{A}$ and we apply Lemma 5.2.1 with $\tau = +\infty$ to write $a_t = \sum_{i \geqslant 1} a_i(t)\mathbf{1}_{E_i}$ and $\nu_t = \sum_{i \geqslant 1} \nu_i(t)\mathbf{1}_{E_i}$ for all $t \geqslant 0$, where $(a_i, \nu_i) \in \mathcal{A}_0$ and $\{E_i,\ i \geqslant 1\} \subset \mathcal{F}_\infty$ form a partition of Ω. Take a set E such that $\mathbb{P}^{\tilde a}_{\tilde\nu}(E) = 0$ for every $(\tilde a, \tilde\nu) \in \mathcal{A}_0$, then

$$\mathbb{P}^a_\nu(E) = \sum_{i \geqslant 1} \mathbb{P}^a_\nu(E \cap E_i) = \sum_{i \geqslant 1} \mathbb{P}^{a_i}_{\nu_i}(E \cap E_i) = 0.$$

(iii) Let N be a $\mathbb{P}^a_\nu$-local martingale, and let us prove by induction that N has a martingale representation property under $\mathbb{P}^a_\nu$, on the interval $[0, \tau_n]$.

As we can choose $\tau_0 = 0$ without loss of generality, the result is trivially true for $n = 0$. Suppose that N has a martingale representation on $[0, \tau_n)$. We apply Lemma 5.2.1 with $\tau = \tau_n$ and $\tilde\tau = \tau_{n+1}$, then $a_t = \sum_{i \geqslant 1} a_i(t)\mathbf{1}_{E_i}$ and $\nu_t = \sum_{i \geqslant 1} \nu^i(t)\mathbf{1}_{E_i}$ for all $\tau_n \leqslant t < \tau_{n+1}$, where $(a_i, \nu^i) \in \mathcal{A}_0$ and $\{E_i,\ i \geqslant 1\} \subset \mathcal{F}_{\tau_n}$ form a partition of Ω. We have that for each $i \geqslant 1$, N^i is a $\mathbb{P}^{a_i}_{\nu^i}$-local martingale, where

$$N^i_t := \left(N_{t \wedge \tau_{n+1}} - N_{\tau_n}\right)\mathbf{1}_{E_i}\mathbf{1}_{[\tau_n, +\infty)}(t).$$

Since $(a_i, \nu^i) \in \mathcal{A}_0$, then by assumption there exist processes H^i and ψ^i such that

$$N^i_t := \int_{\tau_n}^t H^i_s dB^c_s + \int_{\tau_n}^t \int_E \psi^i_s(x) \left(\mu_{B^d}(ds, dx) - \nu^i_s(ds, dx)\right), \ \mathbb{P}^{a_i}_{\nu^i}\text{-a.s}, \ \tau_n \leqslant t < \tau_{n+1}.$$

We define

$$H_t := \sum_{i \geqslant 1} H^i_t \mathbf{1}_{E_i} \ \text{ and } \ \psi_t(x) := \sum_{i \geqslant 1} \psi^i_t(x) \mathbf{1}_{E_i}, \ \forall x \in E, \ \tau_n \leqslant t < \tau_{n+1},$$

then

$$N_t := \int_{\tau_n}^t H_s dB^c_s + \int_{\tau_n}^t \int_E \psi_s(x) \left(\mu_{B^d}(ds, dx) - \nu_s(ds, dx)\right), \ \mathbb{P}^a_\nu\text{-a.s}, \ \tau_n \leqslant t < \tau_{n+1}.$$

So N has a martingale representation on $[0, \tau_{n+1}]$, and the induction is complete. Now recall that $\inf\{n : \tau_n = \infty\} < +\infty$ to conclude that N has a martingale representation on $[0, +\infty)$.

(iv) Take $(a, \nu) \in \mathcal{A}$ of the form (5.2.3), in which we can take $\tau_0 = 0$ without loss of generality.

There exists $0 < t_0 < \tau_1$ such that for every $t \leqslant t_0$, $\mathbb{P}^a_\nu$ is the law on $[0, t_0]$ of a semimartingale with characteristics $\left(-\int_0^t \int_E x \mathbf{1}_{|x|>1} \tilde{\nu}_s(dx) ds, \int_0^t \tilde{a}_s ds, \tilde{\nu}_s(dx) ds\right)$ where

$$\tilde{a}_t := \sum_{i \geqslant 1} a^0_i(t) \mathbf{1}_{E^0_i} \text{ and } \tilde{\nu}_t := \sum_{i \geqslant 1} \nu^0_i(t) \mathbf{1}_{E^0_i},$$

where $\{E^0_i, \, i \geqslant 1\} \subset \mathcal{F}_0$ is a partition of Ω. Since $\mathcal{F}_0$ is trivial, the partition is only composed of Ω and $\emptyset$, and then

$$\tilde{a}_t := a^0_1(t) \text{ and } \tilde{\nu}_t = \nu^0_1(t).$$

Then for $E \in \mathcal{F}_{0^+}$,

$$\mathbb{P}^a_\nu(E) = \mathbb{P}^{\tilde{a}}_{\tilde{\nu}}(E) = 0 \text{ ou } 1,$$

since $\mathbb{P}^{\tilde{a}}_{\tilde{\nu}}$ satisfies the Blumenthal $0-1$ law by hypothesis. □

Remark 5.2.4. *If $\mathcal{A}_0$ consists in deterministic mappings as in example 5.2.1, then $\mathbb{P}^a_\nu$ is the law on $[0, \tau_1]$ of an additive process with non random characteristics, for which the Blumenthal $0-1$ law holds (see for instance [96]).*

We now state the following Proposition which tells us that our probability measure coincides until their first time of disagreement.

Proposition 5.2.2. *Let $\mathcal{A}$ be a separable class of coefficients generated by $\mathcal{A}_0$, let $\mathcal{P}_\mathcal{A} := \{\mathbb{P}^a_\nu, \ (a, \nu) \in \mathcal{A}\}$ and let $(a, \nu^1) \times (b, \nu^2) \in \mathcal{A} \times \mathcal{A}$.*

(i) *$\theta^{a,b}_{\nu^1,\nu^2}$ is an $\mathcal{F}$-stopping time taking countably many values.*

(ii) *Moreover, we have the following coherence condition*

$$\mathbb{P}^a_{\nu^1}\left(E\cap\Omega^{a,\nu^1,b,\nu^2}_{\widehat{\tau}}\right)=\mathbb{P}^b_{\nu^2}\left(E\cap\Omega^{a,\nu^1,b,\nu^2}_{\widehat{\tau}}\right),\ \forall\widehat{\tau}\in\mathcal{T}^{\mathcal{P}_{\mathcal{A}}},\ \forall E\in\mathcal{F}^{\mathcal{P}_{\mathcal{A}}}_{\widehat{\tau}}.$$

Proof.

(i) Let us prove that $\left\{\theta^{a,b}_{\nu^1,\nu^2}\leqslant t_1\right\}\in\mathcal{F}_{t_1}$, for any $t_1\geqslant 0$.

We apply Lemma 5.2.1 for (a,ν^1) and (b,ν^2) with $\tau=t_1$ to obtain that a_t and b_t coincide with $a_i(t)$ and $b_i(t)$ on E_i and that ν^j_t coincides with $\nu^j_i(t)$ on E_i, $j=1,2$, for $t<\tilde{\tau}$, where $\tilde{\tau}>t_1$, $(a_i,\nu^1_i)\times(b_i,\nu^2_i)\in\mathcal{A}_0\times\mathcal{A}_0$ and $\{E_i,\ i\geqslant 1\}\subset\mathcal{F}_{t_1}$ form a partition of Ω. Then

$$\left\{\theta^{a,b}_{\nu_1,\nu_2}\leqslant t_1\right\}=\bigcup_{i\geqslant 1}\left\{\theta^{a_i,b_i}_{\nu^1_i,\nu^2_i}\leqslant t_1\right\}\cap E_i$$

By the constant disagreement times property of $\mathcal{A}_0$, $\left\{\theta^{a_i,b_i}_{\nu^1_i,\nu^2_i}\leqslant t_1\right\}$ is either Ω or $\emptyset$, and since $E_i\in\mathcal{F}_{t_1}$, then

$$\left\{\theta^{a,b}_{\nu^1,\nu^2}\leqslant t_1\right\}\in\mathcal{F}_{t_1}.$$

To show that $\theta^{a,b}_{\nu^1,\nu^2}$ takes countably many values, we apply again Lemma 5.2.1 with $\tau=\theta^{a,b}_{\nu^1,\nu^2}$, which gives that a_t and b_t coincide with $a_i(t)$ and $b_i(t)$ on E_i and that ν^j_t coincides with $\nu^j_i(t)$ on E_i, $j=1,2$, for $t<\tilde{\tau}$, where $\tilde{\tau}>\tau$, $(a_i,\nu^1_i)\times(b_i,\nu^2_i)\in\mathcal{A}_0\times\mathcal{A}_0$ and $\{E_i,\ i\geqslant 1\}\subset\mathcal{F}_\tau$ form a partition of Ω. Since $\theta^{a_i,b_i}_{\nu^1_i,\nu^2_i}$ is a constant and given that $\theta^{a,b}_{\nu^1,\nu^2}=\theta^{a_i,b_i}_{\nu^1_i,\nu^2_i}$ on E_i, we have the desired result.

(ii) We write that

$$\begin{aligned}E\cap\Omega^{a,\nu^1,b,\nu^2}_{\widehat{\tau}}\cap\left\{\theta^{a,b}_{\nu^1,\nu^2}\leqslant t\right\}&=E\cap\left\{\widehat{\tau}<\theta^{a,b}_{\nu^1,\nu^2}\right\}\cap\left\{\theta^{a,b}_{\nu^1,\nu^2}\leqslant t\right\}\\&=\bigcup_{m\geqslant 1}\left(E\cap\left\{\widehat{\tau}<\theta^{a,b}_{\nu^1,\nu^2}\right\}\cap\left\{\widehat{\tau}\leqslant t-\frac{1}{m}\right\}\cap\left\{\theta^{a,b}_{\nu^1,\nu^2}\leqslant t\right\}\right).\end{aligned}$$

Since $\left\{\theta^{a,b}_{\nu^1,\nu^2}\leqslant t\right\}\in\mathcal{F}_t$, we get that for any $m\geqslant 1$,

$$\begin{aligned}E\cap\left\{\widehat{\tau}<\theta^{a,b}_{\nu^1,\nu^2}\right\}\cap\left\{\widehat{\tau}\leqslant t-\frac{1}{m}\right\}\in\mathcal{F}^{\mathcal{P}}_{t-\frac{1}{m}}&\subset\mathcal{F}^{+}_{t-\frac{1}{m}}\vee\mathcal{N}^{\mathbb{P}^a_{\nu_1}}(\mathcal{F}_\infty)\\&\subset\mathcal{F}_t\vee\mathcal{N}^{\mathbb{P}^a_{\nu_1}}(\mathcal{F}_\infty),\end{aligned}$$

and then

$$E\cap\Omega^{a,\nu^1,b,\nu^2}_{\widehat{\tau}}\in\mathcal{F}_{\theta^{a,b}_{\nu^1,\nu^2}}\vee\mathcal{N}^{\mathbb{P}^a_{\nu_1}}(\mathcal{F}_\infty).$$

From this last assertion, we deduce that there exist measurable sets E_i^{a,ν^1}, E_i^{b,ν^2} belonging to $\mathcal{F}_{\theta^{a,b}_{\nu^1,\nu^2}}$, $i=1,2$, such that

$$E_1^{a,\nu^1} \subset E \cap \Omega_{\widehat{\tau}}^{a,\nu^1,b,\nu^2} \subset E_2^{a,\nu^1},\ E_1^{b,\nu^2} \subset E \cap \Omega_{\widehat{\tau}}^{a,\nu^1,b,\nu^2} \subset E_2^{b,\nu^2}$$
$$\mathbb{P}^a_{\nu^1}\left(E_2^{a,\nu^1} \backslash E_1^{a,\nu^1}\right) = \mathbb{P}^b_{\nu^2}\left(E_2^{b,\nu^2} \backslash E_1^{b,\nu^2}\right) = 0.$$

We set $E^1 := E_1^{a,\nu^1} \cup E_1^{b,\nu^2}$ and $E^2 := E_2^{b,\nu^2} \cap E_1^{b,\nu^2}$, then

$$E^1,\, E^2 \in \mathcal{F}_{\theta^{a,b}_{\nu^1,\nu^2}},\ E^1 \subset E \cap \Omega_{\widehat{\tau}}^{a,\nu^1,b,\nu^2} \subset E^2 \text{ and } \mathbb{P}^a_{\nu^1}\left(E^2 \backslash E^1\right) = \mathbb{P}^b_{\nu^2}\left(E^2 \backslash E^1\right) = 0.$$

This implies that

$$\mathbb{P}^a_{\nu^1}\left(E \cap \Omega_{\widehat{\tau}}^{a,\nu^1,b,\nu^2}\right) = \mathbb{P}^a_{\nu^1}(E^2) \text{ and } \mathbb{P}^b_{\nu^2}\left(E \cap \Omega_{\widehat{\tau}}^{a,\nu^1,b,\nu^2}\right) = \mathbb{P}^b_{\nu^2}(E^2),$$

but the solutions of the martingale problems $(\mathbb{P}^0, 0, \theta^{a,b}_{\nu^1,\nu^2}, a, \nu^1)$ and $(\mathbb{P}^0, 0, \theta^{a,b}_{\nu^1,\nu^2}, b, \nu^2)$ are equal by definition. And since $E^2 \in \mathcal{F}_{\theta^{a,b}_{\nu^1,\nu^2}}$, we have

$$\mathbb{P}^a_{\nu^1}(E^2) = \mathbb{P}^b_{\nu^2}(E^2)$$

which gives the desired result. $\square$

We now have all tools we need to state and prove the main result of this section, which generalizes the aggregation result of Theorem 5.1 in [103]. For this purpose, we use the more general aggregation result of Cohen [23], that does not concern only volatility or jump measure uncertainty.

Theorem 5.2.1. *Let $\mathcal{A}$ be a separable class of coefficients generated by $\mathcal{A}_0$ and $\mathcal{P}_{\mathcal{A}}$ the corresponding probability measures. Let*

$$\left\{X^{a,\nu},\ (a,\nu) \in \mathcal{A}\right\},$$

be a family of $\widehat{\mathbb{F}}^{\mathcal{P}_{\mathcal{A}}}$-progressively measurable processes.

Then the following two conditions are equivalent

(i) *$\{X^{a,\nu},\ (a,\nu) \in \mathcal{A}\}$ satisfies the following consistency condition*

$$X^{a,\nu^1} = X^{b,\nu^2},\ \mathbb{P}^a_{\nu^1}\text{-a.s. on } [0, \theta^{a,b}_{\nu^1,\nu^2}) \text{ for any } (a,\nu^1) \in \mathcal{A} \text{ and } (b,\nu^2) \in \mathcal{A}.$$

(ii) *There exists a $\mathcal{P}_{\mathcal{A}}$-q.s. unique process X such that*

$$X = X^{a,\nu},\ \mathbb{P}^a_{\nu}\text{-a.s.},\ \forall (a,\nu) \in \mathcal{A}.$$

Proof. We first prove that (i) implies (ii). Using Lemma 3 in [23], we see that the definition of the generating classes, together with Proposition 5.2.2, implies that the family $\mathcal{P}_{\mathcal{A}}$ satisfies the Hahn property defined in [23]. Now Theorem 4 of [23] gives the result. The fact that (ii) implies (i) is a consequence of the uniqueness of the solution of the martingale problem $(\mathbb{P}^0, 0, +\infty, a, \nu^1)$ on $[0, \theta^{a,b}_{\nu^1,\nu^2})$. □

Now that we have Theorem 5.2.1, we can answer our first issue concerning the aggregation of the predictable compensators associated to the jump measure μ_{B^d} of the canonical process. Indeed, let $\mathcal{A}$ be a separable class of coefficients generated by $\mathcal{A}_0$. Then, for each Borel set $A \in \mathcal{B}(E)$ and for each $t \in [0,T]$ the family $\left\{\nu_t^{\mathbb{P}^a_\nu}(A)\right\}_{(a,\nu)\in\mathcal{A}}$ clearly satisfies the consistency condition above (because it is defined trough the Doob-Meyer decomposition), and therefore there exists a process $\widehat{\nu}$ such that

$$\widehat{\nu}_t(A) = \nu_t^{\mathbb{P}}(A), \text{ for every } \mathbb{P} \in \mathcal{P}_{\mathcal{A}}. \tag{5.2.5}$$

We then denote

$$\widetilde{\mu}_{B^d}(dt,dx) := \mu_{B^d}(dt,dx) - \widehat{\nu}_t(dx)dt.$$

5.2.1.4 The strong formulation

In this subsection, we will concentrate on a subset of $\mathcal{P}_W$. For this purpose, we define

$$\mathcal{V} := \{\nu \in \mathcal{N},\ (I_d,\nu) \in \mathcal{A}_W\}.$$

For each $\nu \in \mathcal{V}$, we denote $\mathbb{P}^\nu := \mathbb{P}^{I_d}_\nu$ and for each $\alpha \in \mathcal{D}$, we define

$$\mathbb{P}^{\alpha,\nu} := \mathbb{P}^\nu \circ (X^\alpha_\cdot)^{-1}, \text{ where } X^\alpha_t := \int_0^t \alpha_s^{1/2} dB^c_s + B^d_t,\ \mathbb{P}^\nu - a.s. \tag{5.2.6}$$

Let us now define,

$$\mathcal{P}_S := \{\mathbb{P}^{\alpha,\nu},\ (\alpha,\nu) \in \mathcal{A}_W\}.$$

Then α is the quadratic variation density of the continuous part of X^α and

$$dB^c_s = \alpha_s^{-1/2} dX^{\alpha,c}_s,$$

under $\mathbb{P}^\nu$. Moreover, $\nu_t(dx)dt$ is the compensator of the measure associated to the jumps of X^α and $\Delta X^\alpha_s = \Delta B_s$ under $\mathbb{P}^\nu$.

We also define for each $\mathbb{P} \in \mathcal{P}_W$ the following process

$$L^{\mathbb{P}}_t := W^{\mathbb{P}}_t + B^d_t,\ \mathbb{P} - a.s., \tag{5.2.7}$$

where $W^{\mathbb{P}}_t$ is a $\mathbb{P}$-Brownian motion defined by

$$W^{\mathbb{P}}_t := \int_0^t \widehat{a}_s^{-1/2} dB^c_s.$$

Then, $\mathcal{P}_S$ is a subset of $\mathcal{P}_W$ and we have by definition

$$\text{the } \mathbb{P}^{\alpha,\nu}\text{-distribution of } (B, \widehat{a}, \widehat{\nu}, L^{\mathbb{P}^{\alpha,\nu}}) \text{ is equal to the } \mathbb{P}^{\nu}\text{-distribution of } (X^{\alpha}, \alpha, \nu, B). \tag{5.2.8}$$

We also have the following characterization in terms of filtrations, which is similar to Lemma 8.1 in [103]

Lemma 5.2.3. $\mathcal{P}_S = \left\{\mathbb{P} \in \mathcal{P}_W,\ \overline{\mathbb{F}^{L^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathbb{F}}^{\mathbb{P}}\right\}$

Proof. By the above remarks, it is clear that α and B are $\overline{\mathbb{F}^{X^{\alpha}}}^{\mathbb{P}^{\nu}}$-progressively measurable. But by definition, $\mathbb{F}$ is generated by B, thus we conclude easily that $\overline{\mathbb{F}}^{\mathbb{P}^{\nu}} \subset \overline{\mathbb{F}^{X^{\alpha}}}^{\mathbb{P}^{\nu}}$. The other inclusion being clear by definition, we have

$$\overline{\mathbb{F}}^{\mathbb{P}^{\nu}} = \overline{\mathbb{F}^{X^{\alpha}}}^{\mathbb{P}^{\nu}}.$$

Now we can use (5.2.8) to obtain that

$$\overline{\mathbb{F}^{L^{\mathbb{P}^{\alpha,\nu}}}}^{\mathbb{P}^{\alpha,\nu}} = \overline{\mathbb{F}}^{\mathbb{P}^{\alpha,\nu}}.$$

Conversely, let $\mathbb{P} \in \mathcal{P}_W$ be such that $\overline{\mathbb{F}^{L^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathbb{F}}^{\mathbb{P}}$. Then, there exists some measurable function β such that $B_{\cdot} = \beta(L^{\mathbb{P}}_{\cdot})$. Let ν be the compensator of the measure associated to the jumps of B under $\mathbb{P}$. Define then,

$$\alpha_t := \frac{d < \beta(B), \beta(B) >^c_t}{dt},$$

we conclude then that $\mathbb{P} = \mathbb{P}^{\alpha,\nu}$. □

Define now $\mathcal{A}_S := \{(\alpha, \nu) \in \mathcal{A}_W,\ \mathbb{P}^{\alpha}_{\nu} \in \mathcal{P}_S\}$. It is important to notice that in our framework, it is not clear whether all the probability measures in $\mathcal{P}_S$ satisfy the martingale representation property and the Blumenthal $0-1$ law. Indeed, this is due to the fact that the process $L^{\mathbb{P}}$ does not necessarily satisfy them. This is a major difference with [103]. Nonetheless, if we restrict ourselves to a subset of $\mathcal{P}_S$, we are going to see that we can still recover them.

First, we have the following generalization of Proposition 8.3 of [103].

Proposition 5.2.3. *Let $\mathcal{A}$ be a separable class of coefficients generated by $\mathcal{A}_0$. If $\mathcal{A}_0 \subset \mathcal{A}_S$, then $\mathcal{A} \subset \mathcal{A}_S$.*

Proof. This is a straightforward generalization of the proof of Proposition 8.3 in [103], using the same kind of modifications as in our previous proofs, so we omit it. □

Let us now consider the set introduced above in Example 5.2.1

$$\tilde{\mathcal{A}}_0 := \{(\alpha, \nu) \in \mathcal{D} \times \mathcal{N} \text{ which are deterministic}\},\ \mathcal{P}_{\tilde{\mathcal{A}}_0} := \left\{\mathbb{P}^{\alpha}_{\nu},\ (\alpha, \nu) \in \tilde{\mathcal{A}}_0\right\}.$$

$\tilde{\mathcal{A}}_0$ is a generating class of coefficients, and it is a well known result that $\tilde{\mathcal{A}}_0 \subset \mathcal{A}_W$ (see Theorem III.2.16 in [56]) and that every probability measure in $\mathcal{P}_{\tilde{\mathcal{A}}_0}$ satisfies the martingale representation property and the Blumenthal $0-1$ law, since the canonical process is actually an additive process under them. Moreover we also have

Lemma 5.2.4. *We have*

$$\mathcal{P}_{\tilde{\mathcal{A}}_0} \subset \mathcal{P}_S.$$

Proof. Let $\mathbb{P} := \mathbb{P}^\alpha_\nu$ be a probability measure in $\mathcal{P}_{\tilde{\mathcal{A}}_0}$. As argued previously, we have $\mathbb{P}-a.s.$

$$B^c_t = \int_0^t \alpha_s^{1/2} dL^{\mathbb{P},c}_t \text{ and } \Delta L^{\mathbb{P}}_t = \Delta B_t.$$

Since α is deterministic, it is clear that we have $\overline{\mathbb{F}}^{\mathbb{P}} = \overline{\mathbb{F}^{L^{\mathbb{P}}}}^{\mathbb{P}}$, which implies the result. □

Finally, we consider $\tilde{\mathcal{A}}$ the separable class of coefficients generated by $\tilde{\mathcal{A}}_0$ and $\mathcal{P}_{\tilde{\mathcal{A}}}$ the corresponding set of probability measures. Then, using the above results and Propositions 5.2.1 and 5.2.3, we have

Proposition 5.2.4. *$\mathcal{P}_{\tilde{\mathcal{A}}} \subset \mathcal{P}_S$ and every probability measure in $\mathcal{P}_{\tilde{\mathcal{A}}}$ satisfies the martingale representation property and the Blumenthal $0-1$ law.*

Proof. Once we know that the augmented filtration generated by $L^{\mathbb{P}}$ satisfies the martingale representation property and the Blumenthal $0-1$ law for every $\mathbb{P} \in \mathcal{P}_{\widetilde{\mathcal{A}}_0}$, we can argue exactly as in the proof of Lemma 8.2 of [103] to obtain the results for $\mathcal{P}_{\widetilde{\mathcal{A}}_0}$. The result for $\mathcal{P}_{\widetilde{\mathcal{A}}}$ then comes easily from Proposition 5.2.1. □

Remark 5.2.5. *In our jump framework, we need to impose this separability structure on both α and ν, in order to be able to retrieve not only the aggregation result of Theorem 5.2.1 but also the property that all our probability measures satisfy the Blumenthal $0-1$ law and the martingale representation property. However, if one is only interested in being able to consider standard BSDEJs, then we do not need the aggregation result and we can work with a larger set of probability measures without restrictions on the α. Namely, let us define*

$$\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}} := \left\{ \mathbb{P}^{a,\nu},\ a \in \mathcal{D},\ (I_d,\nu) \in \widetilde{\mathcal{A}} \right\}.$$

Then we can show as above that $\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}} \subset \mathcal{P}_S$ and that all the probability measures in $\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}}$ satisfy the Blumenthal $0-1$ law and the martingale representation property. This is going to be useful for us in Section 5.4.4.

5.2.2 The nonlinear generator

In this subsection we will introduce the function which will serve as the generator of our 2BSDEJs. Let us define the spaces

$$\widehat{L}^2 := \cap_{\nu \in \mathcal{N}} L^2(\nu) \text{ and } \widehat{L}^1 := \cap_{\nu \in \mathcal{N}} L^1(\nu).$$

For any $\mathcal{C}^1$ function v with bounded gradient, any $\omega \in \Omega$ and any $0 \leqslant t \leqslant T$, we denote $\tilde{v}$ the function

$$\tilde{v}(e) := v(e+\omega(t)) - v(\omega(t)) - \mathbf{1}_{\{|e| \leqslant 1\}}\, e.(\nabla v)(\omega(t)),\ \text{ for } e \in E.$$

The hypothesis on v ensure that $\tilde{v}$ is an element of $\widehat{L}^1$. We then consider a map

$$H_t(\omega, y, z, u, \gamma, \tilde{v}) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \widehat{L}^2 \times D_1 \times D_2 \to \mathbb{R},$$

where $D_1 \subset \mathbb{R}^{d\times d}$ is a given subset containing 0, $D_2 \subset \widehat{L}^1 \cap \mathcal{C}_K(E)$, and $\mathcal{C}_K(E)$ denotes the set of continuous functions on E with a compact support.

Define the following conjugate of H with respect to γ and v by

$$F_t(\omega, y, z, u, a, \nu) := \sup_{\{\gamma,\tilde{v}\}\in D_1\times D_2} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) + < \tilde{v}, \nu > - H_t(\omega, y, z, u, \gamma, \tilde{v}) \right\},$$

for $a \in \mathbb{S}_d^{>0}$ and $\nu \in \mathcal{N}$, and where $< \tilde{v}, \nu >$ is defined by

$$< \tilde{v}, \nu > := \int_E \tilde{v}(e)\nu(de). \tag{5.2.9}$$

The quantity $< \tilde{v}, \nu >$ will not appear again in the chapter, since we formulate the needed hypothesis for the backward equation generator directly on the function F. But the particular form of $< \tilde{v}, \nu >$ comes from the intuition that the 2BSDEJ is an essential supremum of classical BSDEJs. Indeed, solutions to Markovian BSDEJs provide viscosity solutions to some parabolic partial integro-differential equations whose non local operator is given by a quantity similar to $< \tilde{v}, \nu >$ (see [5] for more details).

We define

$$\widehat{F}_t(y, z, u) := F_t(y, z, u, \widehat{a}_t, \widehat{\nu}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0, 0, 0), \ \mathbb{P}^{\alpha,\nu}\text{-a.s.} \tag{5.2.10}$$

We denote by $D^1_{F_t(y,z,u)}$ the domain of F in a and by $D^2_{F_t(y,z,u)}$ the domain of F in ν, for a fixed (t, ω, y, z, u).

As in [101] we fix a constant $\kappa \in (1, 2]$ and restrict the probability measures in $\mathcal{P}_H^\kappa \subset \mathcal{P}_{\tilde{\mathcal{A}}}$

Definition 5.2.6. *$\mathcal{P}_H^\kappa$ consists of all $\mathbb{P} \in \mathcal{P}_{\tilde{\mathcal{A}}}$ such that*

$$\underline{a}^{\mathbb{P}} \leqslant \widehat{a} \leqslant \overline{a}^{\mathbb{P}},\ dt \times d\mathbb{P} - a.e. \text{ for some } \underline{a}^{\mathbb{P}}, \overline{a}^{\mathbb{P}} \in \mathbb{S}_d^{>0}, \text{ and } \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}_t^0\right|^\kappa dt\right)^{\frac{2}{\kappa}}\right] < +\infty,$$

$$\int_E (1 \wedge |x|^2)\underline{\nu}^{\mathbb{P}}(dx) \leqslant \int_E (1 \wedge |x|^2)\widehat{\nu}_t(dx) \leqslant \int_E (1 \wedge |x|^2)\overline{\nu}^{\mathbb{P}}(dx), \text{ and}$$

$$\int_{|x|>1} |x|\, \underline{\nu}^{\mathbb{P}}(dx) \leqslant \int_{|x|>1} |x|\, \widehat{\nu}_t(dx) \leqslant \int_{|x|>1} |x|\, \overline{\nu}^{\mathbb{P}}(dx),\ dt \times d\mathbb{P} - a.e.$$

for $\underline{\nu}^{\mathbb{P}}, \overline{\nu}^{\mathbb{P}}$, two $\sigma-$finite Lévy measures in $\mathcal{N}$.

Remark 5.2.6. *With the above definition, for a fixed $\mathbb{P} \in \mathcal{P}_H^\kappa$, we have*

$$\int_0^T \int_E (1 \wedge |x|^2)\underline{\nu}^{\mathbb{P}}(dx) \leqslant \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_E (1 \wedge |x|^2)\widehat{\nu}_t(dx)\right] \leqslant \int_0^T \int_E (1 \wedge |x|^2)\overline{\nu}^{\mathbb{P}}(dx) < \infty,$$

$$\text{and } \int_0^T \int_{|x|>1} |x|\, \underline{\nu}^{\mathbb{P}}(dx) \leqslant \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{|x|>1} |x|\, \widehat{\nu}_t(dx)\right] \leqslant \int_0^T \int_{|x|>1} |x|\, \overline{\nu}^{\mathbb{P}}(dx) < \infty.$$

We now state our main assumptions on the function F which will be our main interest in the sequel

Assumption 5.2.1. (i) *The domains $D^1_{F_t(y,z,u)} = D^1_{F_t}$ and $D^2_{F_t(y,z,u)} = D^2_{F_t}$ are independent of (ω, y, z, u).*

(ii) *For fixed (y, z, a, ν), F is $\mathbb{F}$-progressively measurable in $D^1_{F_t} \times D^2_{F_t}$.*

(iii) *We have the following uniform Lipschitz-type property in y and z*

$$\forall (y, y^{'}, z, z^{'}, u, t, a, \nu, \omega),$$
$$\left| F_t(\omega, y, z, u, a, \nu) - F_t(\omega, y^{'}, z^{'}, u, a, \nu) \right| \leqslant C \left(\left| y - y^{'} \right| + \left| a^{1/2} \left(z - z^{'} \right) \right| \right).$$

(iv) *For all $(t, \omega, y, z, u^1, u^2, a, \nu)$, there exist two processes γ and $\gamma^{'}$ such that*

$$\int_E \left(u^1(e) - u^2(e) \right) \gamma_t(e) \nu(de) \leqslant F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu),$$

$$F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu) \leqslant \int_E \left(u^1(e) - u^2(e) \right) \gamma^{'}_t(e) \nu(de) \text{ and}$$
$$c_1(1 \wedge |x|) \leqslant \gamma_t(x) \leqslant c_2(1 \wedge |x|) \text{ where } c_1 \leqslant 0,\ 0 \leqslant c_2 < 1,$$
$$c^{'}_1(1 \wedge |x|) \leqslant \gamma^{'}_t(x) \leqslant c^{'}_2(1 \wedge |x|) \text{ where } c^{'}_1 \leqslant 0,\ 0 \leqslant c^{'}_2 < 1.$$

(v) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

Remark 5.2.7. (i) *For $\kappa_1 < \kappa_2$, applying Hölder's inequality gives us*

$$\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T \left| \widehat{F}^0_t \right|^{\kappa_1} dt \right)^{\frac{2}{\kappa_1}} \right] \leqslant C \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T \left| \widehat{F}^0_t \right|^{\kappa_2} dt \right)^{\frac{2}{\kappa_2}} \right],$$

where C is a constant. Then it is clear that $\mathcal{P}^\kappa_H$ is decreasing in κ.

(ii) *The Assumption 5.2.1, together with the fact that $\widehat{F}^0_t < +\infty$, $\mathbb{P}^{\alpha,\nu}$-a.s for every $\mathbb{P}^{\alpha,\nu} \in \mathcal{P}^\kappa_H$, implies that $\widehat{a}_t \in D^1_{F_t}$ and $\widehat{\nu} \in D^2_{F_t}$ $dt \times d\mathbb{P}^{\alpha,\nu}$-a.e., for all $\mathbb{P}^{\alpha,\nu} \in \mathcal{P}^\kappa_H$.*

5.2.3 The spaces and norms

We now define as in [101], the spaces and norms which will be needed for the formulation of the second order BSDEs.

For $p \geqslant 1$, $L^{p,\kappa}_H$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|^p_{L^{p,\kappa}_H} := \sup_{\mathbb{P} \in \mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}} \left[|\xi|^p \right] < +\infty.$$

$\mathbb{H}^{p,\kappa}_H$ denotes the space of all $\mathbb{F}^+$-predictable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|^p_{\mathbb{H}^{p,\kappa}_H} := \sup_{\mathbb{P} \in \mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |\widehat{a}^{1/2}_t Z_t|^2 dt \right)^{\frac{p}{2}} \right] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H^\kappa - q.s. \text{ càdlàg paths, and } \|Y\|_{\mathbb{D}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T} |Y_t|^p\right] < +\infty.$$

$\mathbb{J}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-predictable functions U with

$$\|U\|_{\mathbb{J}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \int_E |U_s(x)|^2 \,\widehat{\nu}_t(dx)ds\right)^{\frac{p}{2}}\right] < +\infty.$$

For each $\xi \in L_H^{1,\kappa}$, $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $t \in [0,T]$ denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}[\xi] \text{ where } \mathcal{P}_H^\kappa(t^+,\mathbb{P}) := \left\{\mathbb{P}' \in \mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\right\}.$$

Then we define for each $p \geqslant \kappa$,

$$\mathbb{L}_H^{p,\kappa} := \left\{\xi \in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < +\infty\right\} \text{ where } \|\xi\|_{\mathbb{L}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leqslant t\leqslant T}{\operatorname{ess\,sup}^{\mathbb{P}}} \left(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right].$$

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}}, \text{ for every } 1 \leqslant \kappa \leqslant p.$$

For a given probability measure $\mathbb{P} \in \mathcal{P}_H^\kappa$, the spaces $L^p(\mathbb{P})$, $\mathbb{D}^p(\mathbb{P})$, $\mathbb{H}^p(\mathbb{P})$ and $\mathbb{J}^p(\mathbb{P})$ correspond to the above spaces when the set of probability measures is only the singleton $\{\mathbb{P}\}$. Finally, we have $\mathbb{H}_{loc}^p(\mathbb{P})$ denotes the space of all $\mathbb{F}^+$-predictable $\mathbb{R}^d$-valued processes Z with

$$\left(\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right)^{\frac{p}{2}} < +\infty, \ \mathbb{P} - a.s.$$

$\mathbb{J}_{loc}^p(\mathbb{P})$ denotes the space of all $\mathbb{F}^+$-predictable functions U with

$$\left(\int_0^T \int_E |U_s(x)|^2 \,\widehat{\nu}_t(dx)ds\right)^{\frac{p}{2}} < +\infty, \ \mathbb{P} - a.s.$$

5.2.4 Formulation

We shall consider the following 2BSDEJ, for $0 \leqslant t \leqslant T$ and $\mathcal{P}_H^\kappa$-q.s.

$$Y_t = \xi - \int_t^T \widehat{F}_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s^c - \int_t^T \int_E U_s(x)\tilde{\mu}_{B^d}(ds, dx) + K_T - K_t. \quad (5.2.11)$$

Definition 5.2.7. *We say $(Y, Z, U) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$ is a solution to 2BSDEJ (5.2.11) if*

- *$Y_T = \xi$, $\mathcal{P}_H^\kappa$-q.s.*
- *For all $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $0 \leqslant t \leqslant T$, the process $K^\mathbb{P}$ defined below is predictable and has nondecreasing paths $\mathbb{P} - a.s.$*

$$K_t^\mathbb{P} := Y_0 - Y_t + \int_0^t \widehat{F}_s(Y_s, Z_s, U_s)ds + \int_0^t Z_s dB_s^c + \int_0^t \int_E U_s(x)\tilde{\mu}_{B^d}(ds, dx). \quad (5.2.12)$$

- *The family $\left\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ satisfies the minimum condition*

$$K_t^\mathbb{P} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,inf}^\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\right], \ 0 \leqslant t \leqslant T, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \quad (5.2.13)$$

Moreover if the family $\left\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ can be aggregated into a universal process K, we call (Y, Z, U, K) a solution of the 2BSDEJ (5.2.11).

Remark 5.2.8. *Since with our set $\mathcal{P}_H^\kappa$ we have the aggregation property of Theorem 5.2.1, and since the minimum condition (5.2.13) implies easily that the family $\left\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ satisfies the consistency condition, we can apply Theorem 5.2.1 and find an aggregator for the family. This is different from [101] or [90], because we are working with a smaller set of probability measures.*

Following [101], in addition to Assumption 5.2.1, we will always assume

Assumption 5.2.2. (i) *$\mathcal{P}_H^\kappa$ is not empty.*

(ii) *The process $\widehat{F}^0$ satisfy the following integrability condition*

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\underset{0 \leqslant t \leqslant T}{\operatorname{ess\,sup}^\mathbb{P}} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\int_0^T |\hat{F}_s^0|^\kappa ds\right]\right)^{\frac{2}{\kappa}}\right] < +\infty \quad (5.2.14)$$

5.2.5 Connection with standard BSDEJs

Let us assume that H is linear in γ and $\tilde{v}$, in the following sense

$$H_t(y, z, u, \gamma, \tilde{v}) := \frac{1}{2}\mathrm{Tr}\left[I_d \gamma\right] + \langle \tilde{v}, \nu^* \rangle - f_t(y, z, u), \quad (5.2.15)$$

where $\nu^* \in \mathcal{N}$. We then have the following result

Lemma 5.2.5. *If H is of the form (5.2.15), then $D_{F_t}^1 = \{I_d\}$, $D_{F_t}^2 = \{\nu^*\}$ and*

$$F_t(\omega, y, z, u, a, \nu) = F_t(\omega, y, z, u, Id, \nu^*) = f_t(y, z, u).$$

Proof. First notice that

$$\begin{aligned} H_t(\omega, y, z, u, \gamma, \tilde{v}) = \sup_{(a,\nu) \in \mathbb{S}_d^{>0} \times \mathcal{N}} &\left\{\frac{1}{2}\mathrm{Tr}(a\gamma) + \int_0^T \int_E \tilde{v}(e)\nu_s(\omega)(ds, de) - \delta_{Id}(a) - \delta_{\nu^*}(\nu)\right\} \\ &- f_t(y, z, u). \end{aligned}$$

By definition of F, we get

$$F_t(\omega, y, z, u, a, \nu) = f_t(y, z, u) + H^{**}(a, \nu),$$

where H^{**} is the double Fenchel-Legendre transform of the function

$$(a, \nu) \mapsto \delta_{Id}(a) + \delta_{\nu^*}(\nu),$$

with $\delta_{Id}(a) = 0\mathbf{1}_{\{a=Id\}} + \infty\mathbf{1}_{\{a\neq Id\}}$ and $\delta_{\nu^*}(\nu) = 0\mathbf{1}_{\{\nu=\nu^*\}} + \infty\mathbf{1}_{\{\nu\neq\nu^*\}}$.

The above function is convex and lower-semicontinuous, implying that

$$F_t(\omega, y, z, u, a, \nu) = f_t(y, z, u) + \delta_{Id}(a) + \delta_{\nu^*}(\nu),$$

which is the desired result. $\square$

If we further assume that $\mathbb{E}^{\mathbb{P}_{\nu^*}}\left[\int_0^T |f_t(0,0,0)|^2 dt\right] < +\infty$, then $\mathcal{P}_H^\kappa = \{\mathbb{P}_{\nu^*}\}$ and the minimality condition on $K = K^{\mathbb{P}_{\nu^*}}$ implies that $0 = \mathbb{E}^{\mathbb{P}_{\nu^*}}[K_T]$, which means that $K = 0$, $\mathbb{P}_{\nu^*}$-a.s. and the 2BSDEJ is reduced to a classical BSDEJ.

5.2.6 Connection with G-expectations and G-Lévy processes

In a recent paper [55], Hu and Peng introduced a new class of processes with independent and stationary increments, called G-Lévy processes. These processes are defined without making reference to any probability measure.

Let $\widetilde{\Omega}$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\widetilde{\Omega}$, containing the constants and such that $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. A sublinear expectation is a functional $\widehat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ which is monotone nondecreasing, constant preserving, sub-additive and positively homogeneous. We refer to Definition 1.1 of [89] for more details. The triple $(\widetilde{\Omega}, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 5.2.8. *A d-dimensional càdlàg process $\{X_t,\, t \geqslant 0\}$ defined on a sublinear expectation space $(\widetilde{\Omega}, \mathcal{H}, \widehat{\mathbb{E}})$ is called a G-Lévy process if:*

(i) $X_0 = 0$.

(ii) *X has independent increments: $\forall s, t > 0$, the random variable $(X_{t+s} - X_t)$ is independent from $(X_{t_1}, \dots, X_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leqslant t_1 < \dots < t_n \leqslant t$. The notion of independence used here corresponds to definition 3.10 in [89].*

(iii) *X has stationary increments: $\forall s, t > 0$, the distribution of $(X_{t+s} - X_t)$ does not depend on t. The notion of distribution used here corresponds to the definition given in §3 of [89].*

(iv) *For each $t \geqslant 0$, there exists a decomposition $X_t = X_t^c + X_t^d$, where $\{X_t^c,\, t \geqslant 0\}$ is a continuous process and $\{X_t^d,\, t \geqslant 0\}$ is a pure jump process.*

(v) *(X_t^c, X_t^d) is a 2d-dimensional process satisfying conditions (i), (ii) and (iii) of this definition and*

$$\lim_{t \to 0^+} \frac{1}{t}\widehat{\mathbb{E}}\left(|X_t^c|^3\right) = 0, \;\; \widehat{\mathbb{E}}\left(\left|X_t^d\right|\right) \leqslant Ct,\, t \geqslant 0$$

for a real constant C.

In [55], Hu and Peng proved the following Lévy-Khintchine representation for G-Lévy processes:

Theorem 5.2.2 ([55]). *Let $\{X_t,\, t \geqslant 0\}$ be a G-Lévy process. Then for each Lipschitz and bounded function φ, the function u defined by $u(t,x) := \widehat{\mathbb{E}}\left(\varphi[x+X_t]\right)$ is the unique viscosity solution of the following partial integro-differential equation:*

$$\partial_t u(t,x) - \sup_{(b,\alpha,\nu)\in\mathcal{U}}\{\int_E [u(t,x+z)-u(t,x)]\nu(dz) \\ + < Du(t,x), b >_{\mathbb{R}^d} + \frac{1}{2}\mathrm{Tr}\left[D^2u(t,x)\alpha\alpha^T\right]\} = 0$$

where $\mathcal{U}$ is a subset of $\mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathcal{M}_R^+$ satisfying

$$\sup_{(b,\alpha,\nu)\in\mathcal{U}}\left\{\int_{\mathbb{R}^d} |z|\,\nu(dz) + |b| + \mathrm{Tr}\left[\alpha\alpha^T\right]\right\} < +\infty$$

and where $\mathcal{M}_R^+$ denotes the set of positive Radon measures on E.

Hu and Peng studied the case of G-Lévy processes with a discontinuous part that is of finite variation. In our framework, we know that B^d is a purely discontinuous semi-martingale of finite variation if $\int_0^T \int_{|x|\leqslant 1} |x|\,\nu_s(dx)ds < +\infty$, $\mathbb{P}_\nu$-a.s. We give a function H below, that is the natural candidate to retrieve the example of G-Lévy processes in our context. This is one of the points of our paper [62].

Let $\tilde{\mathcal{N}}$ be any subset of $\mathcal{N}$ that is convex and closed for the weak topology on $\mathcal{M}_R^+$. We define

$$H_t(\omega,\gamma,\tilde{v}) := \sup_{(a,\nu)\in\mathbb{S}_d^{>0}\times\mathcal{N}}\left\{\frac{1}{2}\mathrm{Tr}(a\gamma) + \int_0^T\int_E \tilde{v}(e)\nu_s(de)ds - \delta_{[a_1,a_2]}(a) - \delta_{\tilde{\mathcal{N}}}(\nu)\right\}.$$

Since $[a_1,a_2]$ and $\tilde{\mathcal{N}}$ are closed convex spaces, $F_t(\omega,a,\nu)$ is the double Fenchel-Legendre transform in (a,ν) of the convex and lower semi-continuous function $(a,\nu)\mapsto \delta_{[a_1,a_2]}(a) + \delta_{\tilde{\mathcal{N}}}(\nu)$ and then

$$F_t(\omega,a,\nu) = \delta_{[a_1,a_2]}(a) + \delta_{\tilde{\mathcal{N}}}(\nu),$$

where $\delta_{[a_1,a_2]}(a) = 0\mathbf{1}_{\{a\in[a_1,a_2]\}} + \infty\mathbf{1}_{\{a\notin[a_1,a_2]\}}$ and $\delta_{\tilde{\mathcal{N}}}(\nu) = 0\mathbf{1}_{\{\nu\in\tilde{\mathcal{N}}\}} + \infty\mathbf{1}_{\{\nu\notin\tilde{\mathcal{N}}\}}$.

5.3 Uniqueness result

5.3.1 Representation of the solution

We have similarly as in Theorem 4.4 of [101]

Theorem 5.3.1. *Let Assumptions 5.2.1 and 5.2.2 hold. Assume $\xi \in \mathbb{L}_H^{2,\kappa}$ and that (Y,Z,U) is a solution to 2BSDEJ (5.2.11). Then, for any $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $0 \leqslant t_1 < t_2 \leqslant T$,*

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \ \mathbb{P}-a.s., \tag{5.3.1}$$

where, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}^+$-stopping time τ, and $\mathcal{F}_\tau^+$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, $(y^{\mathbb{P}}(\tau,\xi), z^{\mathbb{P}}(\tau,\xi))$ denotes the solution to the following standard BSDE on $0 \leqslant t \leqslant \tau$

$$y_t^{\mathbb{P}} = \xi - \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, u_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s^c - \int_t^\tau \int_E u_s^{\mathbb{P}}(x)\tilde{\mu}_{B^d}(ds,\ dx),\ \mathbb{P}-a.s. \quad (5.3.2)$$

Remark 5.3.1. *We first emphasize that existence and uniqueness results for the standard BSDEs (5.3.2) are not given directly by the existing literature, since the compensator of the counting measure associated to the jumps of B is not deterministic. However, since all the probability measures we consider satisfy the martingale representation property and the Blumenthal $0-1$ law, it is clear that we can straightforwardly generalize the proof of existence and uniqueness of Tang and Li [106] (see also [8] and [24] for related results). Furthermore, the usual a priori estimates and comparison Theorems will also hold.*

Remark 5.3.2. *It is worth noticing that, unlike in the case of 2BSDEs (see [101] for example), this representation does not imply directly the uniqueness of the solution in $\mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$.*

Indeed, by taking $t_2 = T$ in this representation formula, we have

$$Y_t = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{}^{\mathbb{P}}\ y_t^{\mathbb{P}'}(T,\xi),\ t \in [0,T],\ \mathbb{P}-a.s.,\ \textit{for all } \mathbb{P} \in \mathcal{P}_H^\kappa,$$

and thus Y is unique.

Then, since we have that $d\langle Y^c, B^c\rangle_t = Z_t d\langle B^c\rangle_t$, $\mathcal{P}_H^\kappa - q.s.$, Z is unique. However, here we are not able to obtain that U and $K^{\mathbb{P}}$ are uniquely determined. Nonetheless, this representation is necessary to prove some a priori estimates in Theorem 5.3.4 which, as for the standard BSDEJs, insure the uniqueness of the solution.

Before giving the proof of the above theorem, we first state the following Lemma which is a generalization of the comparison theorem proved by Royer (see Theorem 2.5 in [95]). Its proof is a straightforward generalization so we omit it.

Lemma 5.3.1. *Let $\mathbb{P} \in \mathcal{P}_H^\kappa$. We consider two generators f^1 and f^2 satisfying Assumption H_{comp} in [95] (which is a consequence of our more restrictive Assumption 5.2.1(iv)). Given two nondecreasing processes k^1 and k^2, let ξ^1 and ξ^2 be two terminal conditions for the following BSDEJs driven respectively by f^1 and f^2,*

$$\begin{aligned} y_t^i =& \xi^i - \int_t^T f_s^i(y_s^i, z_s^i, u_s^i)ds - \int_t^T z_s^i dB_s^c - \int_t^T \int_E u_s^i(x)\tilde{\mu}_{B^d}(ds,\ dx) \\ &+ k_T^i - k_t^i,\ \textit{for } i = 1,2, \quad \mathbb{P}-a.s. \end{aligned}$$

Denote by (y^1, z^1, u^1) and (y^2, z^2, u^2) the respective solutions. If $\xi^1 \leqslant \xi^2$, $k^1 - k^2$ is non-increasing and $f^1(t, y_t^1, z_t^1, u_t^1) \geqslant f^2(t, y_t^1, z_t^1, u_t^1)$, then $\forall t \in [0,T], y_t^1 \leqslant y_t^2$.

Proof. [Proof of Theorem 5.3.1] The proof follows the lines of the proof of Theorem 4.4 in [101].

(i) Fix $0 \leqslant t_1 < t_2 \leqslant T$ and $\mathbb{P} \in \mathcal{P}_H^\kappa$. For any $\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})$ and $t_1 \leqslant t \leqslant t_2$, we have,

$$\begin{aligned} Y_t =& Y_{t_2} - \int_t^{t_2} \widehat{F}_s(Y_s, Z_s, U_s)ds - \int_t^{t_2} Z_s dB_s^c - \int_t^{t_2}\int_E U_s(x)\tilde{\mu}_{B^d}(ds, dx) \\ &+ K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'},\ \mathbb{P}' - a.s. \end{aligned}$$

With Assumption 5.2.1, we can apply the above Lemma 5.3.1 under $\mathbb{P}'$ to obtain $Y_{t_1} \geqslant y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P}' - a.s.$. Since $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_{t_1}^+$, we get $Y_{t_1} \geqslant y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$ and thus

$$Y_{t_1} \geqslant \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_H^\kappa$. We will show in (iii) below that

$$C_{t_1}^{\mathbb{P}} := \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^2\right] < +\infty,\ \mathbb{P} - a.s.$$

For every $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$, denote

$$\delta Y := Y - y^{\mathbb{P}'}(t_2, Y_{t_2}),\ \delta Z := Z - z^{\mathbb{P}'}(t_2, Y_{t_2}) \text{ and } \delta U := U - u^{\mathbb{P}'}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption 5.2.1(iii), there exist two bounded processes λ and η such that for all $t_1 \leqslant t \leqslant t_2$,

$$\begin{aligned} \delta Y_t =& \int_t^{t_2} \left(\lambda_s \delta Y_s + \eta_s \widehat{a}_s^{1/2} \delta Z_s\right) ds - \int_t^{t_2} \left(\widehat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}, U_s) - \widehat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}, u_s^{\mathbb{P}'})\right) ds \\ &- \int_t^{t_2} \delta Z_s dB_s^c - \int_t^{t_2}\int_E \delta U_s(x)\tilde{\mu}_{B^d}(ds, dx) + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'},\ \mathbb{P}' - a.s. \end{aligned}$$

Define for $t_1 \leqslant t \leqslant t_2$ the following processes

$$N_t := \int_{t_1}^t \eta_s \widehat{a}_s^{-1/2} dB_s^c - \int_{t_1}^t \int_E \gamma_s(x)\tilde{\mu}_{B^d}(ds,\ dx),$$

and

$$M_t := \exp\left(\int_{t_1}^t \lambda_s ds\right)\mathcal{E}(N)_t,$$

where $\mathcal{E}(N)_t$ denotes the Doléans-Dade exponential martingale of N_t.

By the boundedness of λ and η and the assumption on γ in Assumption 5.2.1(iv), we know that M has moments (positive or negative) of any order (see [72] for the positive moments and Lemma 5.6.6 in the Appendix for the negative ones). Thus we have for $p \geqslant 1$

$$\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\sup_{t_1 \leqslant t \leqslant t_2} M_t^p + \sup_{t_1 \leqslant t \leqslant t_2} M_t^{-p}\right] \leqslant C_p,\ \mathbb{P}' - a.s. \tag{5.3.3}$$

Then, by Itô's formula, we obtain

$$\begin{aligned}
d(M_t\delta Y_t) =& M_{t-}d(\delta Y_t) + \delta Y_{t-}dM_t + d\left[M,\delta Y\right]_t \\
=& M_{t-}\left[\left(-\lambda_t\delta Y_t - \eta_t\widehat{a}_t^{1/2}\delta Z_t + \widehat{F}_t(y_t^{\mathbb{P}'}, z_t^{\mathbb{P}'}, U_t) - \widehat{F}_t(y_t^{\mathbb{P}'}, z_t^{\mathbb{P}'}, u_t^{\mathbb{P}'})\right)dt\right. \\
& \left. + \delta Z_t dB_t^c + \int_E \left(\delta U_t(x) - \gamma_t(x)\delta U_t(x)\right)\tilde{\mu}_{B^d}(dt,dx)\right] \\
& + \delta Y_{t-}M_{t-}\left(\lambda_t dt + \eta_t\widehat{a}_t^{-1/2}dB_t^c - \int_E \gamma_t(x)\tilde{\mu}_{B^d}(dt,dx)\right) \\
& + M_t\left(\eta_t\widehat{a}_t^{1/2}\delta Z_t dt - \int_E \gamma_t(x)\delta U_t(x)\widehat{\nu}_t(dx)dt\right) - M_{t-}dK_t^{\mathbb{P}'}.
\end{aligned}$$

Thus, by Assumption 5.2.1(iv), we have

$$\begin{aligned}
\delta Y_{t_1} \leqslant & -\int_{t_1}^{t_2} M_s\left(\delta Z_s + \delta Y_s\eta_s\widehat{a}_s^{-1/2}\right)dB_s^c + \int_{t_1}^{t_2} M_{s-}dK_s^{\mathbb{P}'} \\
& - \int_{t_1}^{t_2} M_{s-}\int_E \left(\delta U_s(x) - \delta Y_s\gamma_s(x) - \gamma_s(x)\delta U_s(x)\right)\tilde{\mu}_{B^d}(ds,dx).
\end{aligned}$$

By taking conditional expectation, we obtain

$$\delta Y_{t_1} \leqslant \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\int_{t_1}^{t_2} M_{t-}dK_t^{\mathbb{P}'}\right]. \tag{5.3.4}$$

Applying the Hölder inequality, we can now write

$$\begin{aligned}
\delta Y_{t_1} &\leqslant \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\sup_{t_1\leqslant t\leqslant t_2}(M_t)\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)\right] \\
&\leqslant \left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\sup_{t_1\leqslant t\leqslant t_2}(M_t)^3\right]\right)^{1/3}\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^{3/2}\right]\right)^{2/3} \\
&\leqslant C(C_{t_1}^{\mathbb{P}})^{1/3}\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right]\right)^{1/3}, \ \mathbb{P}-a.s.
\end{aligned}$$

Taking the essential infimum on both sides finishes the proof.

(iii) It remains to show that the estimate for $C_{t_1}^{\mathbb{P}}$ holds. But by definition, and the Lipschitz Assumption on F we clearly have

$$\begin{aligned}
\sup_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^2\right] &\leqslant C\left(\|Y\|_{\mathbb{D}_H^{2,\kappa}}^2 + \|Z\|_{\mathcal{H}_H^{2,\kappa}}^2 + \|U\|_{\mathbb{J}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa}\right) \\
&< +\infty,
\end{aligned} \tag{5.3.5}$$

since the last term on the right-hand side is finite thanks to the integrability assumed on ξ and $\widehat{F}^0$.

We then use the definition of the essential supremum (see Neveu [85] for example) to have the following equality

$$\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})} \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^2\right] = \sup_{n\geqslant 1}\mathbb{E}_{t_1}^{\mathbb{P}_n}\left[\left(K_{t_2}^{\mathbb{P}_n} - K_{t_1}^{\mathbb{P}_n}\right)^2\right], \ \mathbb{P}-a.s. \tag{5.3.6}$$

for some sequence $(\mathbb{P}_n)_{n\geqslant 1}\subset\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$.

Moreover, in Lemma 5.6.3 of the Appendix, it is proved that the set $\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$ is upward directed which means that for any $\mathbb{P}_1^{'}$, $\mathbb{P}_2^{'}\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$, there exists $\mathbb{P}^{'}\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$ such that

$$\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\left(K_{t_2}^{\mathbb{P}^{'}}-K_{t_1}^{\mathbb{P}^{'}}\right)^2\right]=\max\left\{\mathbb{E}_{t_1}^{\mathbb{P}_1^{'}}\left[\left(\mathrm{K}_{t_2}^{\mathbb{P}_1^{'}}-\mathrm{K}_{t_1}^{\mathbb{P}_1^{'}}\right)^2\right],\ \mathbb{E}_{t_1}^{\mathbb{P}_2^{'}}\left[\left(\mathrm{K}_{t_2}^{\mathbb{P}_2^{'}}-\mathrm{K}_{t_1}^{\mathbb{P}_2^{'}}\right)^2\right]\right\}.$$

Hence, by using a subsequence if necessary, we can rewrite (5.3.6) as

$$\underset{\mathbb{P}^{'}\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\left(K_{t_2}^{\mathbb{P}^{'}}-K_{t_1}^{\mathbb{P}^{'}}\right)^2\right]=\lim_{n\to\infty}\uparrow\mathbb{E}_{t_1}^{\mathbb{P}_n}\left[\left(K_{t_2}^{\mathbb{P}_n}-K_{t_1}^{\mathbb{P}_n}\right)^2\right],\ \mathbb{P}-a.s.$$

With (5.3.5), we can then finish the proof exactly as in the proof of Theorem 4.4 in [101]. ◻

Finally, the comparison Theorem below follows easily from the classical one for BSDEJs (see for instance Theorem 2.5 in [95]) and the representation (5.3.1).

Theorem 5.3.2. *Let (Y,Z,U) and (Y',Z',U') be the solutions of 2BSDEJs with terminal conditions ξ and $\xi^{'}$, generators $\widehat{F}$ and $\widehat{F}^{'}$ respectively, and let $(y^{\mathbb{P}},z^{\mathbb{P}},u^{\mathbb{P}})$ and $(y'^{\mathbb{P}},z'^{\mathbb{P}},u'^{\mathbb{P}})$ the solutions of the associated BSDEJs. Assume that they both verify our Assumptions 5.2.1 and 5.2.2 and that we have*

- $\xi\leqslant\xi^{'}$, $\mathcal{P}_H^\kappa-q.s.$
- $\widehat{F}_t(y_t'^{\mathbb{P}},z_t'^{\mathbb{P}},u_t'^{\mathbb{P}})\geqslant\widehat{F}_t^{'}(y_t'^{\mathbb{P}},z_t'^{\mathbb{P}},u_t'^{\mathbb{P}})$, $\mathbb{P}-a.s.$, *for all* $\mathbb{P}\in\mathcal{P}_H^\kappa$.

Then $Y\leqslant Y'$, $\mathcal{P}_H^\kappa-q.s.$

5.3.2 A priori estimates and uniqueness of the solution

We conclude this section by showing some *a priori* estimates which not only will imply uniqueness of the solution of the 2BSDEJ (5.2.11), but also will be useful to obtain the existence of a solution.

Theorem 5.3.3. *Let Assumptions 5.2.1 and 5.2.2 hold. Assume $\xi\in\mathbb{L}_H^{2,\kappa}$ and $(Y,Z,U)\in\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}\times\mathbb{J}_H^{2,\kappa}$ is a solution to the 2BSDEJ (5.2.11). Let $\left\{(y^{\mathbb{P}},z^{\mathbb{P}},u^{\mathbb{P}})\right\}_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ be the solutions of the corresponding BSDEJs (5.3.2). Then, there exists a constant C_κ depending only on κ, T and the Lipschitz constant of F such that*

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}}+\|Z\|^2_{\mathbb{H}_H^{2,\kappa}}+\|U\|^2_{\mathbb{J}_H^{2,\kappa}}+\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left|K_T^{\mathbb{P}}\right|^2\right]\leqslant C_\kappa\left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}}+\phi_H^{2,\kappa}\right),$$

and

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\left\{\left\|y^{\mathbb{P}}\right\|^2_{\mathbb{D}^2(\mathbb{P})}+\left\|z^{\mathbb{P}}\right\|^2_{\mathbb{H}^2(\mathbb{P})}+\left\|u^{\mathbb{P}}\right\|^2_{\mathbb{J}^2(\mathbb{P})}\right\}\leqslant C_\kappa\left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}}+\phi_H^{2,\kappa}\right).$$

Proof. As in the proof of the representation formula in Theorem 5.3.1, the Lipschitz assumption 5.2.1(iii) of F implies that there exist two bounded processes λ and η such that for all t,

$$\begin{aligned} y_t^{\mathbb{P}} =& \xi + \int_t^T \left(\lambda_s y_s^{\mathbb{P}} + \eta_s \widehat{a}_s^{1/2} z_s^{\mathbb{P}}\right) ds - \int_t^T \left(\widehat{F}_s(0,0,u_s^{\mathbb{P}})\right) ds \\ & - \int_t^T z_s^{\mathbb{P}} dB_s^c - \int_t^T \int_E u_s^{\mathbb{P}}(x) \tilde{\mu}_{B^d}(ds,dx), \ \mathbb{P}-\text{a.s.} \end{aligned}$$

Define the following processes

$$N_r := \int_t^r \eta_s \widehat{a}_s^{-1/2} dB_s^c - \int_t^r \int_E \gamma_s(x) \tilde{\mu}_{B^d}(ds,dx),$$

and

$$M_r := \exp\left(\int_t^r \lambda_s ds\right) \mathcal{E}(N)_r,$$

where $\mathcal{E}(N)_r$ denotes the Doléans-Dade exponential martingale of N_r.

Then by applying Itô's formula to $M_t y_t^{\mathbb{P}}$, we obtain

$$y_t^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{P}} \left[M_T \xi - \int_t^T M_s \widehat{F}_s(0,0,u_s^{\mathbb{P}}) ds + \int_t^T \int_E M_s \gamma_s(x) u_s^{\mathbb{P}}(x) \widehat{\nu}_s(dx) ds \right]$$

Finally with Assumption (5.2.1)(iv), the Hölder inequality and the inequality (5.3.3), we conclude that there exists a constant C_κ depending only on κ, T and the Lipschitz constant of F, such that for all $\mathbb{P}$

$$\left|y_t^{\mathbb{P}}\right| \leqslant C_\kappa \mathbb{E}_t^{\mathbb{P}} \left[|\xi|^\kappa + \int_t^T \left|\widehat{F}_s^0\right|^\kappa ds \right]^{1/\kappa}. \tag{5.3.7}$$

This immediately provides the estimate for $y^{\mathbb{P}}$. Now by definition of our norms, we get from (5.3.7) and the representation formula (5.3.1) that

$$\|Y\|_{\mathbb{D}_H^{2,\kappa}}^2 \leqslant C_\kappa \left(\|\xi\|_{\mathbb{L}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa} \right). \tag{5.3.8}$$

Now apply Itô's formula to $|Y|^2$ under each $\mathbb{P} \in \mathcal{P}_H^\kappa$. We get as usual for every $\varepsilon > 0$

$$\begin{aligned} & |Y_0|^2 + \int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt + \int_0^T \int_E |U_t(x)|^2 \widehat{\nu}_t(dx) dt \\ & = |\xi|^2 - 2\int_0^T Y_t \widehat{F}_t(Y_t,Z_t,U_t) dt + 2\int_0^T Y_{t^-} dK_t^{\mathbb{P}} \\ & \quad - 2\int_0^T Y_t Z_t dB_t^c - \int_0^T \int_E \left(|U_t(x)|^2 + 2Y_{t^-} U_t(x)\right) \tilde{\mu}_{B^d}(dt,dx) \\ & \leqslant |\xi|^2 + 2\int_0^T |Y_t| \, |\widehat{F}_t(Y_t,Z_t,U_t)| dt + 2 \sup_{0\leqslant t\leqslant T} |Y_t| \, K_T^{\mathbb{P}} \\ & \quad - 2\int_0^T Y_t Z_t dB_t^c - \int_0^T \int_E \left(|U_t(x)|^2 + 2Y_{t^-} U_t(x)\right) \tilde{\mu}_{B^d}(dt,dx) \end{aligned}$$

By our assumptions on F, we have

$$\left|\widehat{F}_t(Y_t, Z_t, U_t)\right| \leqslant C\left(|Y_t| + \left|\widehat{a}_t^{1/2} Z_t\right| + \left|\widehat{F}_t^0\right| + \left(\int_E |U_t(x)|^2\, \widehat{\nu}_t(dx)\right)^{1/2}\right).$$

With the usual inequality $2ab \leqslant \frac{1}{\varepsilon}a^2 + \varepsilon b^2, \forall \varepsilon > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt + \int_0^T \int_E |U_t(x)|^2\, \widehat{\nu}_t(dx)dt\right] \\
&\leqslant C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \int_0^T |Y_t| \left(\left|\widehat{F}_t^0\right| + |Y_t| + \left|\widehat{a}_t^{1/2} Z_t\right| + \left(\int_E |U_t(x)|^2\, \widehat{\nu}_t(dx)\right)^{1/2}\right) dt\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_{t-}|\, dK_t^{\mathbb{P}}\right] \\
&\leqslant C\left(\|\xi\|_{\mathbb{L}^{2,\kappa}_H} + \mathbb{E}^{\mathbb{P}}\left[\left(1 + \frac{C}{\varepsilon}\right)\sup_{0\leqslant t\leqslant T}|Y_t|^2 + \left(\int_0^T \left|\widehat{F}_t^0\right| dt\right)^2\right]\right) \\
&\quad + \varepsilon\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt + \int_0^T \int_E |U_t(x)|^2\, \widehat{\nu}_t(dx)dt + \left|K_T^{\mathbb{P}}\right|^2\right].
\end{aligned} \tag{5.3.9}$$

Then by definition of our 2BSDEJ, we easily have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\left|K_T^{\mathbb{P}}\right|^2\right] \leqslant C_0\mathbb{E}^{\mathbb{P}}\Bigg[&|\xi|^2 + \sup_{0\leqslant t\leqslant T}|Y_t|^2 + \int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt \\
&+ \int_0^T \int_E |U_t(x)|^2\, \widehat{\nu}_t(dx)dt + \left(\int_0^T \left|\widehat{F}_t^0\right| dt\right)^2\Bigg],
\end{aligned} \tag{5.3.10}$$

for some constant C_0, independent of ε.

Now set $\varepsilon := (2(1+C_0))^{-1}$ and plug (5.3.10) in (5.3.9). One then gets

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt + \int_0^T \int_E |U_t(x)|^2\, \widehat{\nu}_t(dx)dt\right] \leqslant C\mathbb{E}^{\mathbb{P}}\Bigg[&|\xi|^2 + \sup_{0\leqslant t\leqslant T}|Y_t|^2 \\
&+ \left(\int_0^T \left|\widehat{F}_t^0\right| dt\right)^2\Bigg].
\end{aligned}$$

From this and the estimate for Y, we immediately obtain

$$\|Z\|_{\mathbb{H}^{2,\kappa}_H} + \|U\|_{\mathbb{J}^{2,\kappa}_H} \leqslant C\left(\|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi_H^{2,\kappa}\right).$$

Then the estimate for $K^{\mathbb{P}}$ comes from (5.3.10). The estimates for $z^{\mathbb{P}}$ and $u^{\mathbb{P}}$ can be proved similarly. □

Theorem 5.3.4. *Let Assumptions 5.2.1 and 5.2.2 hold. For $i = 1, 2$, let us consider the solutions $(Y^i, Z^i, U^i, \{K^{\mathbb{P},i}, \mathbb{P} \in \mathcal{P}_H^\kappa\})$ of the 2BSDEJs (5.2.11) with terminal condition ξ^i.*

Then, there exists a constant C_κ depending only on κ, T and the Lipschitz constant of F such that

$$\left\|Y^1 - Y^2\right\|_{\mathbb{D}_H^{2,\kappa}} \leqslant C_\kappa \left\|\xi^1 - \xi^2\right\|_{\mathbb{L}_H^{2,\kappa}},$$

and

$$\begin{aligned}
&\left\|Z^1 - Z^2\right\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}\left|K_t^{\mathbb{P},1} - K_t^{\mathbb{P},2}\right|^2\right] + \left\|U^1 - U^2\right\|^2_{\mathbb{J}_H^{2,\kappa}} \\
&\leqslant C_\kappa \left\|\xi^1 - \xi^2\right\|_{\mathbb{L}_H^{2,\kappa}} \left(\left\|\xi^1\right\|_{\mathbb{L}_H^{2,\kappa}} + \left\|\xi^2\right\|_{\mathbb{L}_H^{2,\kappa}} + (\phi_H^{2,\kappa})^{1/2}\right).
\end{aligned}$$

Consequently, the 2BSDEJ (5.2.11) has at most one solution in $\mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$.

Proof. As in the previous Theorem, we can obtain that there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$, such that for all $\mathbb{P}$

$$\left|y_t^{\mathbb{P},1} - y_t^{\mathbb{P},2}\right| \leqslant C_\kappa \mathbb{E}_t^{\mathbb{P}}\left[\left|\xi^1 - \xi^2\right|^\kappa\right]^{1/\kappa}. \tag{5.3.11}$$

Now by definition of our norms, we get from (5.3.11) and the representation formula (5.3.1) that

$$\left\|Y^1 - Y^2\right\|^2_{\mathbb{D}_H^{2,\kappa}} \leqslant C_\kappa \left\|\xi^1 - \xi^2\right\|^2_{\mathbb{L}_H^{2,\kappa}}. \tag{5.3.12}$$

Applying Itô's formula to $|Y^1 - Y^2|^2$, under each $\mathbb{P} \in \mathcal{P}_H^\kappa$, leads to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2}(Z_t^1 - Z_t^2)\right|^2 dt + \int_0^T \int_E \left|U_t^1(x) - U_t^2(x)\right|^2 \widehat{\nu}_t(dx)dt\right] \\
&\leqslant C\mathbb{E}^{\mathbb{P}}\left[\left|\xi^1 - \xi^2\right|^2\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|Y_t^1 - Y_t^2\right| d(K_t^{\mathbb{P},1} - K_t^{\mathbb{P},2})\right] \\
&\quad + C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T \left|Y_t^1 - Y_t^2\right| \Big(\left|Y_t^1 - Y_t^2\right| + |\widehat{a}_t^{1/2}(Z_t^1 - Z_t^2)| \\
&\quad + \left(\int_E \left|U_t^1(x) - U_t^2(x)\right|^2 \widehat{\nu}_t(dx)dt\right)^{1/2}\Big)\, dt\Big] \\
&\leqslant C\left(\left\|\xi^1 - \xi^2\right\|^2_{\mathbb{L}_H^{2,\kappa}} + \left\|Y^1 - Y^2\right\|^2_{\mathbb{D}_H^{2,\kappa}}\right) \\
&\quad + \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2}(Z_t^1 - Z_t^2)\right|^2 dt + \int_0^T \int_E \left|U_t^1(x) - U_t^2(x)\right|^2 \widehat{\nu}_t(dx)dt\right] \\
&\quad + C\left\|Y^1 - Y^2\right\|_{\mathbb{D}_H^{2,\kappa}} \left(\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^2 \left(K_T^i\right)^2\right]\right)^{1/2}.
\end{aligned}$$

The estimates for $(Z^1 - Z^2)$ and $(U^1 - U^2)$ are now obvious from the above inequality and the estimates of Theorem 5.3.3.

Finally the estimate for the difference of the nondecreasing processes is obvious by definition. □

5.4 A direct existence argument

In the article [101], the main tool to prove existence of a solution is the so called regular conditional probability distributions (r.c.p.d.) of Stroock and Varadhan [104]. Indeed, these tools allow to give a pathwise construction for conditional expectations. Since, at least when the generator is null, the y component of the solution of a BSDE can be written as a conditional expectation, the r.c.p.d. allows us to construct solutions of BSDEs pathwise. Earlier in the chapter, we have identified a candidate solution to the 2BSDEJ as an essential supremum of solutions of classical BSDEJs (see (5.3.1)). However those BSDEJs are written under mutually singular probability measures. Hence, being able to construct them pathwise allows us to avoid the problems related to negligible-sets. In this section we will generalize the approach of [101] to the jump case.

5.4.1 Notations

For the convenience of the reader, we recall below some of the notations introduced in [101] and [30]. Remember that we are working in the Skorohod space $\Omega = \mathbb{D}\left([0,T],\mathbb{R}^d\right)$ endowed with the Skorohod metric which makes it a separable space.

- For $0 \leqslant t \leqslant T$, we denote by $\Omega^t := \left\{\omega \in \mathbb{D}\left([t,T],\mathbb{R}^d\right)\right\}$ the shifted canonical space of càdlàg paths on $[t,T]$ which are null at t, B^t the shifted canonical process. Let $\mathcal{N}^t$ be the set of measures ν on $\mathcal{B}(E)$ satisfying

$$\int_t^T \int_E (1 \wedge |x|^2)\nu_s(dx)ds < +\infty \text{ and } \int_t^T \int_{|x|>1} |x|\, \nu_s(dx)ds < +\infty,\ \forall \tilde{\omega} \in \Omega^t, \tag{5.4.1}$$

  and let $\mathcal{D}^t$ be the set of $\mathbb{F}^t$-progressively measurable processes α taking values in $\mathbb{S}_d^{>0}$ with $\int_t^T |\alpha_s|ds < +\infty$, for every $\tilde{\omega} \in \Omega^t$.

  $\mathbb{F}^t$ is the filtration generated by B^t. We define similarly the continuous part of B^t, denoted $B^{t,c}$, its discontinuous part denoted $B^{t,d}$, the density of the quadratic variation of $B^{t,c}$, denoted $\widehat{a}^t$, and $\mu_{B^{t,d}}$ the counting measure associated to the jumps of B^t.

  Exactly as in Section 5.2, we can define semimartingale problems and the corresponding probability measures. We then restrict ourselves to deterministic (α,ν) and we let $\tilde{\mathcal{A}}^t$ be the corresponding separable class of coefficients and $\mathcal{P}_{\tilde{\mathcal{A}}^t}$ the corresponding family of probability measures, which will be noted $\mathbb{P}^{t,\alpha,\nu}$. Then, this family also satisfies the aggregation property of Theorem 5.2.1, and we can define $\widehat{\nu}^t$, the aggregator of the predictable compensators of B^t.

- For $0 \leqslant s \leqslant t \leqslant T$ and $\omega \in \Omega^s$, we define the shifted path $\omega^t \in \Omega^t$ by

$$\omega_r^t := \omega_r - \omega_t,\ \forall r \in [t,T].$$

- For $0 \leqslant s \leqslant t \leqslant T$ and $\omega \in \Omega^s$, $\widetilde{\omega} \in \Omega^t$ we define the concatenation path $\omega \otimes_t \widetilde{\omega} \in \Omega^s$ by
$$(\omega \otimes_t \widetilde{\omega})(r) := \omega_r 1_{[s,t)}(r) + (\omega_t + \widetilde{\omega}_r) 1_{[t,T]}(r), \ \forall r \in [s,T].$$

- For $0 \leqslant s \leqslant t \leqslant T$ and a $\mathcal{F}^s_T$-measurable random variable ξ on Ω^s, for each $\omega \in \Omega^s$, we define the shifted $\mathcal{F}^t_T$-measurable random variable $\xi^{t,\omega}$ on Ω^t by
$$\xi^{t,\omega}(\widetilde{\omega}) := \xi(\omega \otimes_t \widetilde{\omega}), \ \forall \, \widetilde{\omega} \in \Omega^t.$$
Similarly, for an $\mathbb{F}^s$-progressively measurable process X on $[s,T]$ and $(t,\omega) \in [s,T] \times \Omega^s$, we can define the shifted process $\{X^{t,\omega}_r, r \in [t,T]\}$, which is $\mathbb{F}^t$-progressively measurable.

- For a $\mathbb{F}$-stopping time τ, we use the same simplification as [101]
$$\omega \otimes_\tau \widetilde{\omega} := \omega \otimes_{\tau(\omega)} \widetilde{\omega}, \ \xi^{\tau,\omega} := \xi^{\tau(\omega),\omega}, \ X^{\tau,\omega} := X^{\tau(\omega),\omega}.$$

- We define our "shifted" generator
$$\widehat{F}^{t,\omega}_s(\widetilde{\omega}, y, z, u) := F_s(\omega \otimes_t \widetilde{\omega}, y, z, u, \widehat{a}^t_s(\widetilde{\omega}), \widehat{\nu}^t_s(\widetilde{\omega})), \ \forall (s, \widetilde{\omega}) \in [t,T] \times \Omega^t.$$
Then note that since F is assumed to be uniformly continuous in ω under the $\mathbb{L}^\infty$ norm, then so is $\widehat{F}^{t,\omega}$. Notice that this implies that for any $\mathbb{P} \in \mathcal{P}_{\tilde{\mathcal{A}}^t}$
$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left|\widehat{F}^{t,\omega}_s(0,0,0)\right|^\kappa ds\right)^{\frac{2}{\kappa}}\right] < +\infty,$$
for some ω if and only if it holds for all $\omega \in \Omega$.

- Finally, we extend Definition 5.2.6 in the shifted spaces

  Definition 5.4.1. *$\mathcal{P}^{t,\kappa}_H$ consists of all $\mathbb{P} := \mathbb{P}^{t,\alpha,\nu} \in \mathcal{P}_{\tilde{\mathcal{A}}^t}$ such that*
$$\underline{a}^{\mathbb{P}} \leqslant \widehat{a}^t \leqslant \overline{a}^{\mathbb{P}}, \ ds \times d\mathbb{P} - a.e. \ on \ [t,T] \times \Omega^t \ for \ some \ \underline{a}^{\mathbb{P}}, \overline{a}^{\mathbb{P}} \in \mathbb{S}^{>0}_d,$$
$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left|\widehat{F}^{t,\omega}_s(0,0,0)\right|^\kappa ds\right)^{\frac{2}{\kappa}}\right] < +\infty, \ for \ all \ \omega \in \Omega.$$
$$\int_E (1 \wedge |x|^2) \underline{\nu}^{\mathbb{P}}(dx) \leqslant \int_E (1 \wedge |x|^2) \widehat{\nu}^t_s(dx) \leqslant \int_E (1 \wedge |x|^2) \overline{\nu}^{\mathbb{P}}(dx), \ and$$
$$\int_{|x|>1} |x| \, \underline{\nu}^{\mathbb{P}}(dx) \leqslant \int_{|x|>1} |x| \, \widehat{\nu}^t_s(dx) \leqslant \int_{|x|>1} |x| \, \overline{\nu}^{\mathbb{P}}(dx), \ ds \times d\mathbb{P} - a.e.$$
  on $[t,T] \times \Omega^t$ for $\underline{\nu}^{\mathbb{P}}, \overline{\nu}^{\mathbb{P}}$, two $\sigma-$finite Lévy measures in $\mathcal{N}^t$.

Remark 5.4.1. *With the above definition, for a fixed $\mathbb{P} \in \mathcal{P}^{t,\kappa}_H$, we have*
$$\int_t^T \int_E (1 \wedge |x|^2) \underline{\nu}^{\mathbb{P}}(dx) \leqslant \mathbb{E}^{\mathbb{P}}\left[\int_t^T \int_E (1 \wedge |x|^2) \widehat{\nu}^t_s(dx)\right] \leqslant \int_t^T \int_E (1 \wedge |x|^2) \overline{\nu}^{\mathbb{P}}(dx) < \infty,$$
$$and \int_t^T \int_{|x|>1} |x| \, \underline{\nu}^{\mathbb{P}}(dx) \leqslant \mathbb{E}^{\mathbb{P}}\left[\int_t^T \int_{|x|>1} |x| \, \widehat{\nu}^t_s(dx)\right] \leqslant \int_t^T \int_{|x|>1} |x| \, \overline{\nu}^{\mathbb{P}}(dx) < \infty.$$

For given $\omega \in \Omega$, $\mathbb{F}$-stopping time τ and $\mathbb{P} \in \mathcal{P}_H^\kappa$, the r.c.p.d. of $\mathbb{P}$ is a probability measure $\mathbb{P}_\tau^\omega$ on $\mathcal{F}_T$ such that for every bounded $\mathcal{F}_T$-measurable random variable ξ

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\xi\right](\omega) = \mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi], \text{ for } \mathbb{P}\text{-a.e. } \omega.$$

Furthermore, $\mathbb{P}_\tau^\omega$ naturally induces a probability measure $\mathbb{P}^{\tau,\omega}$ on $\mathcal{F}_T^{\tau(\omega)}$ such that the $\mathbb{P}^{\tau,\omega}$-distribution of $B^{\tau(\omega)}$ is equal to the $\mathbb{P}_\tau^\omega$-distribution of $\left\{B_t - B_{\tau(\omega)},\ t \in [\tau(\omega), t]\right\}$. Besides, we have

$$\mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}].$$

Remark 5.4.2. *We emphasize that the above notations correspond to the ones used in [101] when we consider the subset of Ω consisting of all continuous paths from $[0,T]$ to $\mathbb{R}^d$ whose value at time 0 is 0.*

We now prove the following Proposition which gives a relation between $(\widehat{a}^{t,\omega}, \widehat{\nu}^{t,\omega})$ and $(\widehat{a}^t, \widehat{\nu}^t)$.

Proposition 5.4.1. *Let $\mathbb{P} \in \mathcal{P}_H^\kappa$ and τ be an $\mathbb{F}$-stopping time. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have for $ds \times d\mathbb{P}^{\tau,\omega}$-a.e. $(s, \widetilde{\omega}) \in [\tau(\omega), T] \times \Omega^{\tau(\omega)}$*

$$\begin{aligned} &\widehat{a}_s^{\tau,\omega}(\widetilde{\omega}) = \widehat{a}_s^{\tau(\omega)}(\widetilde{\omega}) \\ &\widehat{\nu}_s^{\tau,\omega}(\widetilde{\omega}, A) = \widehat{\nu}_s^{\tau(\omega)}(\widetilde{\omega}, A) \text{ for every } A \in \mathcal{B}(E). \end{aligned}$$

This result is important for us, because it implies that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for $ds \times d\mathbb{P}^{t,\omega} - a.e.$ $(s, \widetilde{\omega}) \in [t, T] \times \Omega^t$

$$F_s\left(\omega \otimes_t \widetilde{\omega}, y, z, u, \widehat{a}_s(\omega \otimes_t \widetilde{\omega}), \widehat{\nu}_s(\omega \otimes_t \widetilde{\omega})\right) = F_s\left(\omega \otimes_t \widetilde{\omega}, y, z, u, \widehat{a}_s^t(\widetilde{\omega}), \widehat{\nu}_s^t(\widetilde{\omega})\right).$$

Whereas the left-hand side has in general no regularity in ω, the right-hand side, that we choose as our shifted generator, is uniformly continuous in ω.

Proof. The proof of the equality for $\widehat{a}$ is the same as the one in Lemma 4.1 of [102], so we omit it.

Now, for $s \geqslant \tau$ and for any $A \in \mathcal{B}(E)$, we know by the Doob-Meyer decomposition that there exist a $\mathbb{P}$-local martingale M and a $\mathbb{P}^{\tau,\omega}$-martingale N such that

$$\mu_{B^d}([0,s], A) = M_s + \int_0^s \widehat{\nu}_r(A)dr,\ \mathbb{P} - a.s.,$$

and

$$\mu_{B^{\tau(\omega),d}}([\tau(\omega), s], A) = N_s + \int_\tau^s \widehat{\nu}_r^{\tau(\omega)}(A)dr.$$

Then, we can rewrite the first equation above for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for $\mathbb{P}^{\tau,\omega}$-a.e. $\widetilde{\omega} \in \Omega^{\tau(\omega)}$

$$\mu_{B^d}(\omega \otimes_\tau \widetilde{\omega}, [0,s], A) = M_s^{\tau,\omega}(\widetilde{\omega}) + \int_0^s \widehat{\nu}_r^{\tau,\omega}(\widetilde{\omega}, A)dr. \tag{5.4.2}$$

Now, by definition of the measures μ_{B^d} and $\mu_{B^{\tau(\omega),d}}$, we have

$$\mu_{B^d}(\omega \otimes_\tau \widetilde{\omega}, [0,s], A) = \mu_{B^d}(\omega, [0,\tau], A) + \mu_{B^{\tau(\omega),d}}(\widetilde{\omega}, [\tau, s], A).$$

Hence, we obtain from (5.4.2) that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for $\mathbb{P}^{\tau,\omega}$-a.e. $\widetilde{\omega} \in \Omega^{\tau(\omega)}$

$$\mu_{B^d}(\omega, [0,\tau], A) - \int_0^\tau \widehat{\nu}_r(\omega, A)dr + N_s(\widetilde{\omega}) - M_s^{\tau,\omega}(\widetilde{\omega}) = \int_\tau^s \left(\widehat{\nu}_r^{\tau,\omega}(\widetilde{\omega}, A) - \widehat{\nu}_r^{\tau(\omega)}(\widetilde{\omega}, A)\right) dr$$

In the left-hand side above, the terms which are $\mathcal{F}_\tau$-measurable are constants in $\Omega^{\tau(\omega)}$ and using the same arguments as in Step 1 of the proof of Lemma 5.6.1, we can show that $M^{\tau,\omega}$ is a $\mathbb{P}^{\tau,\omega}$-local martingale for $\mathbb{P}$-a.e. $\omega \in \Omega$. This means that the left-hand side is a $\mathbb{P}^{\tau,\omega}$-local martingale while the right-hand side is a predictable finite variation process. By the martingale representation property which still holds in the shifted canonical spaces, we deduce that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for $ds \times d\mathbb{P}^{\tau,\omega}$-a.e. $(s, \widetilde{\omega}) \in [\tau(\omega), T] \times \Omega^{\tau(\omega)}$

$$\int_\tau^s \left(\widehat{\nu}_r^{\tau,\omega}(\widetilde{\omega}, A) - \widehat{\nu}_r^{\tau(\omega)}(\widetilde{\omega}, A)\right) dr = 0,$$

which is the desired result. □

5.4.2 Existence when ξ is in $\mathrm{UC_b}(\Omega)$

When ξ is in $\mathrm{UC_b}(\Omega)$, we know that there exists a modulus of continuity function ρ for ξ and F in ω. Then, for any $0 \leqslant t \leqslant s \leqslant T$, $(y, z, \nu) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{V}$ and $\omega, \omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$\left|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})\right| \leqslant \rho\left(\|\omega - \omega'\|_t\right), \ \left|\widehat{F}_s^{t,\omega}(\tilde{\omega}, y, z, u) - \widehat{F}_s^{t,\omega'}(\tilde{\omega}, y, z, u)\right| \leqslant \rho\left(\|\omega - \omega'\|_t\right)$$

We then define for all $\omega \in \Omega$

$$\Lambda(\omega) := \sup_{0 \leqslant s \leqslant t} \Lambda_t(\omega), \tag{5.4.3}$$

where

$$\Lambda_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_H^t} \left(\mathbb{E}^{\mathbb{P}}\left[\left|\xi^{t,\omega}\right|^2 + \int_t^T |\widehat{F}_s^{t,\omega}(0,0,0)|^2 ds\right]\right)^{1/2}.$$

Now since $\widehat{F}^{t,\omega}$ is also uniformly continuous in ω, it is easily verified that

$$\Lambda(\omega) < \infty \text{ for some } \omega \in \Omega \text{ iff it holds for all } \omega \in \Omega. \tag{5.4.4}$$

Moreover, when Λ is finite, it is uniformly continuous in ω under the $\mathbb{L}^\infty$-norm and is therefore $\mathcal{F}_T$-measurable.

Now, by Assumption 5.2.2, we have

$$\Lambda_t(\omega) < \infty \text{ for all } (t, \omega) \in [0,T] \times \Omega. \tag{5.4.5}$$

To prove existence, we define the following value process V_t pathwise

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}_H^{t,\kappa}} \mathcal{Y}_t^{\mathbb{P},t,\omega}(T,\xi), \text{ for all } (t,\omega)\in[0,T]\times\Omega, \tag{5.4.6}$$

where, for any $(t_1,\omega)\in[0,T]\times\Omega$, $\mathbb{P}\in\mathcal{P}_H^{t_1,\kappa}, t_2\in[t_1,T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta\in\mathbb{L}^2(\mathbb{P})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,\eta) := y_{t_1}^{\mathbb{P},t_1,\omega}$, where $\left(y^{\mathbb{P},t_1,\omega}, z^{\mathbb{P},t_1,\omega}, u^{\mathbb{P},t_1,\omega}\right)$ is the solution of the following BSDEJ on the shifted space Ω^{t_1} under $\mathbb{P}$

$$\begin{aligned} y_s^{\mathbb{P},t_1,\omega} &= \eta^{t_1,\omega} - \int_s^{t_2} \widehat{F}_r^{t_1,\omega}\left(y_r^{\mathbb{P},t_1,\omega}, z_r^{\mathbb{P},t_1,\omega},\nu\right)dr - \int_s^{t_2} z_r^{\mathbb{P},t_1,\omega}dB_r^{t_1,c} \\ &\quad - \int_s^{t_2}\int_{\mathbb{R}^d} u_s^{\mathbb{P},t_1,\omega}(x)\widetilde{\mu}_{B^{t_1,d}}(ds,dx),\ \mathbb{P}-a.s.,\ s\in[t,T], \end{aligned} \tag{5.4.7}$$

where as usual $\widetilde{\mu}_{B^{t_1,d}}(ds,dx) := \mu_{B^{t_1,d}}(ds,dx) - \widehat{\nu}_s^{t_1}(dx)ds$.

In view of the Blumenthal $0-1$ law, $y_t^{\mathbb{P},t,\omega}$ is constant for any given (t,ω) and $\mathbb{P}\in\mathcal{P}_H^{t,\kappa}$, and therefore the value process V is well defined. Let us now show that V inherits some properties from ξ and F.

Lemma 5.4.1. *Let Assumptions 5.2.1 and 5.2.2 hold and consider some ξ in $\mathrm{UC_b}(\Omega)$. Then for all $(t,\omega)\in[0,T]\times\Omega$ we have $|V_t(\omega)|\leqslant C\Lambda_t(\omega)$. Moreover, for all $(t,\omega,\omega')\in[0,T]\times\Omega^2$, $|V_t(\omega)-V_t(\omega')|\leqslant C\rho(\|\omega-\omega'\|_t)$. Consequently, V_t is $\mathcal{F}_t$-measurable for every $t\in[0,T]$.*

Proof. (i) For each $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{P}\in\mathcal{P}_H^{t,\kappa}$, let α be some positive constant which will be fixed later and let $\eta\in(0,1)$. Since F is uniformly Lipschitz in (y,z) and satisfies Assumption 5.2.1(iv), we have

$$\left|\widehat{F}_s^{t,\omega}(y,z,u)\right| \leqslant \left|\widehat{F}_s^{t,\omega}(0,0,0)\right| + C\left(|y| + |\left(\widehat{a}_s^t\right)^{1/2}z| + \left(\int_E |u(x)|^2\,\widehat{\nu}_s^t(dx)\right)^{1/2}\right).$$

Now apply Itô's formula. We obtain

$$\begin{aligned} & e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}\right|^2 + \int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2 ds + \int_t^T\int_E e^{\alpha s}\left|u_s^{\mathbb{P},t,\omega}(x)\right|^2\widehat{\nu}_s^t(dx)ds \\ &= e^{\alpha T}\left|\xi^{t,\omega}\right|^2 - 2\int_t^T e^{\alpha s}y_s^{\mathbb{P},t,\omega}\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}, u_s^{\mathbb{P},t,\omega})ds \\ &\quad - \alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2 ds - 2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^{t,c} \\ &\quad - \int_t^T\int_E e^{\alpha s}\left(2y_{s^-}^{\mathbb{P},t,\omega}u_s^{\mathbb{P},t,\omega}(x) + \left|u_s^{\mathbb{P},t,\omega}(x)\right|^2\right)\widetilde{\mu}_{B^{t,d}}(ds,dx) \\ &\leqslant e^{\alpha T}\left|\xi^{t,\omega}\right|^2 + \int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(0,0,0)\right|^2 ds + \left(1+2C+\frac{2C^2}{\eta}-\alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2 ds \\ &\quad + \eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2 ds + \eta\int_t^T\int_E e^{\alpha s}\left|u_s^{\mathbb{P},t,\omega}(x)\right|^2\widehat{\nu}_s^t(dx)ds \\ &\quad - 2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^{t,c} - \int_t^T\int_E e^{\alpha s}\left(2y_{s^-}^{\mathbb{P},t,\omega}u_s^{\mathbb{P},t,\omega}(x) + \left|u_s^{\mathbb{P},t,\omega}(x)\right|^2\right)\widetilde{\mu}_{B^{t,d}}(ds,dx). \end{aligned}$$

Now choose $\eta = 1/2$ for instance and α large enough. By taking expectation we obtain easily

$$\left|y_t^{\mathbb{P},t,\omega}\right|^2 \leqslant C\left|\Lambda_t(\omega)\right|^2.$$

The result then follows from the arbitrariness of $\mathbb{P}$.

(ii) The proof is exactly the same as above, except that one has to use uniform continuity in ω of $\xi^{t,\omega}$ and $\widehat{F}^{t,\omega}$. Indeed, for each $(t,\omega) \in [0,T]\times\Omega$ and $\mathbb{P} \in \mathcal{P}_H^{t,\kappa}$, let α be some positive constant which will be fixed later and let $\eta \in (0,1)$. By Itô's formula we have, since $\widehat{F}$ is uniformly Lipschitz

$$\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega} - y_t^{\mathbb{P},t,\omega'}\right|^2 + \int_t^T e^{\alpha s}\left(\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})\right|^2 + \int_E e^{\alpha s}(u_s^{\mathbb{P},t,\omega} - u_s^{\mathbb{P},t,\omega'})^2(x)\widehat{\nu}_s^t(dx)\right)ds \\
&\leqslant e^{\alpha T}\left|\xi^{t,\omega} - \xi^{t,\omega'}\right|^2 + 2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}\right|^2 ds \\
&+ 2C\int_t^T \left|y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}\right|\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})\right| ds \\
&\quad + 2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}\right|\left(\int_{\mathbb{R}^d}\left|u_s^{\mathbb{P},t,\omega}(x) - u_s^{\mathbb{P},t,\omega'}(x)\right|^2\widehat{\nu}_s^t(dx)\right)^{1/2} ds \\
&\quad + 2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}\right|\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}, u_s^{\mathbb{P},t,\omega}) - \widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}, u_s^{\mathbb{P},t,\omega})\right| ds \\
&\quad - \alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}\right|^2 ds - 2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})dB_s^{t,c} \\
&\quad - \int_t^T\int_E e^{\alpha s}\left(2(y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'})(u_s^{\mathbb{P},t,\omega} - u_s^{\mathbb{P},t,\omega'}) + (u_s^{\mathbb{P},t,\omega} - u_s^{\mathbb{P},t,\omega'})^2\right)(x)\widetilde{\mu}_{B^{t,d}}(ds,dx).
\end{aligned}$$

We then deduce

$$\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega} - y_t^{\mathbb{P},t,\omega'}\right|^2 + \int_t^T e^{\alpha s}\left(\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})\right|^2 + \int_E e^{\alpha s}(u_s^{\mathbb{P},t,\omega} - u_s^{\mathbb{P},t,\omega'})^2(x)\widehat{\nu}_s^t(dx)\right)ds \\
&\leqslant e^{\alpha T}\left|\xi^{t,\omega} - \xi^{t,\omega'}\right|^2 + \int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}, u_s^{\mathbb{P},t,\omega}) - \widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}, u_s^{\mathbb{P},t,\omega})\right|^2 ds \\
&\quad + \eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})\right|^2 ds + \eta\int_t^T\int_E e^{\alpha s}\left|u_s^{\mathbb{P},t,\omega}(x) - u_s^{\mathbb{P},t,\omega'}(x)\right|^2\widehat{\nu}_s^t(dx)ds \\
&\quad + \left(2C + C^2 + \frac{2C^2}{\eta} - \alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}\right|^2 ds \\
&\quad - 2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})dB_s^{t,c} \\
&\quad - \int_t^T\int_E e^{\alpha s}\left(2(y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'})(u_s^{\mathbb{P},t,\omega} - u_s^{\mathbb{P},t,\omega'}) + (u_s^{\mathbb{P},t,\omega} - u_s^{\mathbb{P},t,\omega'})^2\right)(x)\widetilde{\mu}_{B^{t,d}}(ds,dx).
\end{aligned}$$

Now choose $\eta = 1/2$ and α such that $\nu := \alpha - 2C - C^2 - \frac{2C^2}{\eta} \geqslant 0$. We obtain the desired result by taking expectation and using the uniform continuity in ω of ξ and F. □

The next proposition is a dynamic programming property verified by the value process, which will prove useful when proving that V provides a solution to the 2BSDEJ with generator F and terminal condition ξ. The result and its proof are intimately connected and use the same arguments as the proof of Proposition 4.7 in [102].

Proposition 5.4.2. *Under Assumptions 5.2.1, 5.2.2 and for $\xi \in \mathrm{UC_b}(\Omega)$, we have for all $0 \leqslant t_1 < t_2 \leqslant T$ and for all $\omega \in \Omega$*

$$V_{t_1}(\omega) = \sup_{\mathbb{P} \in \mathcal{P}_H^{t_1,\kappa}} \mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2, V_{t_2}^{t_1,\omega}).$$

The proof is almost the same as the proof in [102], with minor modifications due to the introduction of jumps.

Proof. Without loss of generality, we can assume that $t_1 = 0$ and $t_2 = t$. Thus, we have to prove

$$V_0(\omega) = \sup_{\mathbb{P} \in \mathcal{P}_H^{\kappa}} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

Denote $(y^{\mathbb{P}}, z^{\mathbb{P}}, u^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(T,\xi), \mathcal{Z}^{\mathbb{P}}(T,\xi), \mathcal{U}^{\mathbb{P}}(T,\xi))$

(i) For any $\mathbb{P} \in \mathcal{P}_H^{\kappa}$, we know by Lemma 5.6.1 in the Appendix, that for $\mathbb{P} - a.e.\ \omega \in \Omega$, the r.c.p.d. $\mathbb{P}^{t,\omega} \in \mathcal{P}_H^{t,\kappa}$. Now thanks to the paper of Tang and Li [106], we know that the solution of BSDEs on the Wiener-Poisson space with Lipschitz generators can be constructed via Picard iteration. Thus, it means that at each step of the iteration, the solution can be formulated as a conditional expectation under $\mathbb{P}$. By the properties of the r.p.c.d., this entails that

$$y_t^{\mathbb{P}}(\omega) = \mathcal{Y}_t^{\mathbb{P}^{t,\omega},t,\omega}(T,\xi), \text{ for } \mathbb{P} - a.e.\ \omega \in \Omega. \tag{5.4.8}$$

Hence, by definition of V_t and the comparison principle for BSDEJs, we get that $y_0^{\mathbb{P}} \leqslant \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$. By arbitrariness of $\mathbb{P}$, this leads to

$$V_0(\omega) \leqslant \sup_{\mathbb{P} \in \mathcal{P}_H^{\kappa}} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

(ii) For the other inequality, we proceed as in [102]. Let $\mathbb{P} \in \mathcal{P}_H^{\kappa}$ and $\varepsilon > 0$. By separability of Ω, there exists a partition $(E_t^i)_{i \geqslant 1} \subset \mathcal{F}_t$ such that $d_S(\omega, \omega')_t \leqslant \varepsilon/2$ for any i and any $\omega, \omega' \in E_t^i$. Now by Billinsgley [10], we know that the distance for the uniform topology is dominated by the Skorohod metric in the sense that

$$\|\omega - \omega'\|_t \leqslant 2d_S(\omega, \omega')_t \leqslant \varepsilon, \text{ for any } i \text{ and any } \omega, \omega' \in E_t^i. \tag{5.4.9}$$

Now for each i, fix a $\widehat{\omega}_i \in E_t^i$ and let $\mathbb{P}_t^i$ be an $\varepsilon-$optimizer of $V_t(\widehat{\omega}_i)$. If we define for each $n \geqslant 1$, $\mathbb{P}^n := \mathbb{P}^{n,\varepsilon}$ by

$$\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{n} \mathbb{E}^{\mathbb{P}_t^i}\left[1_E^{t,\omega}\right] 1_{E_t^i}\right] + \mathbb{P}(E \cap \widehat{E}_t^n), \text{ where } \widehat{E}_t^n := \cup_{i>n} E_t^i, \tag{5.4.10}$$

then, by Lemma 5.6.2, we know that $\mathbb{P}^n \in \mathcal{P}_H^\kappa$. Besides, by Lemma 5.4.1 and its proof, we have for any i and any $\omega \in E_t^i$

$$\begin{aligned}V_t(\omega) &\leqslant V_t(\widehat{\omega}_i) + C\rho(\varepsilon) \leqslant \mathcal{Y}_t^{\mathbb{P}_t^i,t,\widehat{\omega}_i}(T,\xi) + \varepsilon + C\rho(\varepsilon)\\ &\leqslant \mathcal{Y}_t^{\mathbb{P}_t^i,t,\omega}(T,\xi) + \varepsilon + C\rho(\varepsilon) = \mathcal{Y}_t^{(\mathbb{P}^n)^{t,\omega},t,\omega}(T,\xi) + \varepsilon + C\rho(\varepsilon),\end{aligned}$$

where we used successively the uniform continuity of V in ω and (5.4.9), the definition of $\mathbb{P}_t^i$, the uniform continuity of $\mathcal{Y}_t^{\mathbb{P},t,\omega}$ in ω and finally the definition of $\mathbb{P}^n$.

Then, it follows from (5.4.8) that

$$V_t \leqslant y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon),\ \mathbb{P}^n - a.s. \text{ on } \cup_{i=1}^n E_t^i. \tag{5.4.11}$$

Let now $(y^n, z^n, u^n) := (y^{n,\varepsilon}, z^{n,\varepsilon}, u^{n,\varepsilon})$ be the solution of the following BSDEJ on $[0,t]$

$$\begin{aligned}y_s^n = &\left[y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon)\right] 1_{\cup_{i=1}^n E_t^i} + V_t 1_{\widehat{E}_t^n} - \int_s^t \widehat{F}_r(y_r^n, z_r^n, u_r^n)dr - \int_s^t z_r^n dB_r^c\\ &- \int_s^t \int_E u_r^n(x)\widetilde{\mu}_{B^d}(dr,dx),\ \mathbb{P}^n - a.s.\end{aligned} \tag{5.4.12}$$

By the comparison principle for BSDEJs, we know that $\mathcal{Y}_0^{\mathbb{P}}(t,V_t) \leqslant y_0^n$. Then since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, the equality (5.4.12) also holds $\mathbb{P} - a.s.$ Using the same arguments and notations as in the proof of Lemma 5.4.1, we obtain

$$\left|y_0^n - y_0^{\mathbb{P}^n}\right|^2 \leqslant C\mathbb{E}^{\mathbb{P}}\left[\varepsilon^2 + \rho(\varepsilon)^2 + \left|V_t - y_t^{\mathbb{P}^n}\right|^2 1_{\widehat{E}_t^n}\right].$$

Then, by Lemma 5.4.1, we have

$$\begin{aligned}\mathcal{Y}_0^{\mathbb{P}}(t,V_t) \leqslant y_0^n \leqslant y_0^{\mathbb{P}^n} + C\left(\varepsilon + \rho(\varepsilon) + \left(\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^2 1_{\widehat{E}_t^n}\right]\right)^{1/2}\right)\\ \leqslant V_0(\omega) + C\left(\varepsilon + \rho(\varepsilon) + \left(\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^2 1_{\widehat{E}_t^n}\right]\right)^{1/2}\right).\end{aligned}$$

Then it suffices to let n go to $+\infty$, use the dominated convergence theorem, and finally let ε go to 0. $\square$

Now we are facing the problem of the regularity in t of V. Indeed, if we want to obtain a solution of the 2BSDE, then it has to be at least càdlàg, $\mathcal{P}_H^\kappa - q.s.$ To this end, we define now for all (t,ω), the $\mathbb{F}^+$-progressively measurable process

$$V_t^+ := \overline{\lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}} V_r.$$

Lemma 5.4.2. *Under the conditions of the previous Proposition, we have*

$$V_t^+ = \lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t} V_r,\ \mathcal{P}_H^\kappa - q.s.$$

and thus V^+ is càdlàg $\mathcal{P}_H^\kappa - q.s.$

Proof. For each $\mathbb{P}$, we define

$$\widetilde{V}^{\mathbb{P}} := V - \mathcal{Y}^{\mathbb{P}}(T,\xi).$$

Then, we recall that we have $\widetilde{V}^{\mathbb{P}} \geqslant 0$, $\mathbb{P}-a.s.$ Now for any $0 \leqslant t_1 < t_2 \leqslant T$, let $(y^{\mathbb{P},t_2}, z^{\mathbb{P},t_2}, u^{\mathbb{P},t_2}) := (\mathcal{Y}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{Z}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{U}^{\mathbb{P}}(t_2, V_{t_2}))$. Once more, we remind that since solutions of BSDEs can be defined by Picard iterations, we have by the properties of the r.p.c.d. that

$$\mathcal{Y}^{\mathbb{P}}_{t_1}(t_2, V_{t_2})(\omega) = \mathcal{Y}^{\mathbb{P}^{t_1,\omega},t_1,\omega}_{t_1}(t_2, V^{t_1,\omega}_{t_2}), \text{ for } \mathbb{P}-a.e.\ \omega.$$

Hence, we conclude from Proposition 5.4.2

$$V_{t_1} \geqslant y^{\mathbb{P},t_2}_{t_1},\ \mathbb{P}-a.s.$$

Denote

$$\widetilde{y}^{\mathbb{P},t_2}_t := y^{\mathbb{P},t_2}_t - \mathcal{Y}^{\mathbb{P}}_t(T,\xi),\ \widetilde{z}^{\mathbb{P},t_2}_t := \widehat{a}_t^{-1/2}(z^{\mathbb{P},t_2}_t - \mathcal{Z}^{\mathbb{P}}_t(T,\xi)),\ \widetilde{u}^{\mathbb{P},t_2}_t := u^{\mathbb{P},t_2}_t - \mathcal{U}^{\mathbb{P}}_t(T,\xi).$$

Then $\widetilde{V}^{\mathbb{P}}_{t_1} \geqslant \widetilde{y}^{\mathbb{P},t_2}_{t_1}$ and $(\widetilde{y}^{\mathbb{P},t_2}, \widetilde{z}^{\mathbb{P},t_2}, \widetilde{u}^{\mathbb{P},t_2})$ satisfies the following BSDEJ on $[0,t_2]$

$$\widetilde{y}^{\mathbb{P},t_2}_t = \widetilde{V}^{\mathbb{P}}_{t_2} - \int_t^{t_2} f^{\mathbb{P}}_s(\widetilde{y}^{\mathbb{P},t_2}_s, \widetilde{z}^{\mathbb{P},t_2}_s, \widetilde{u}^{\mathbb{P},t_2}_s)ds - \int_t^{t_2} \widetilde{z}^{\mathbb{P},t_2}_s dW^{\mathbb{P}}_s - \int_t^{t_2}\int_{\mathbb{R}^d} \widetilde{u}^{\mathbb{P},t_2}_s(x)\widetilde{\mu}_{B^d}(ds,dx),$$

where

$$\begin{aligned} f^{\mathbb{P}}_t(\omega,y,z,u) :&= \widehat{F}_t(\omega, y+\mathcal{Y}^{\mathbb{P}}_t(\omega), \widehat{a}_t^{-1/2}(\omega)(z+\mathcal{Z}^{\mathbb{P}}_t(\omega)), u+\bar{\mathcal{U}}^{\mathbb{P}}_t(\omega)) \\ &\quad - \widehat{F}_t(\omega, \mathcal{Y}^{\mathbb{P}}_t(\omega), \mathcal{Z}^{\mathbb{P}}_t(\omega), \mathcal{U}^{\mathbb{P}}_t(\omega)). \end{aligned}$$

By the definition given in Royer [95], we conclude from the above that $\widetilde{V}^{\mathbb{P}}$ is a positive $f^{\mathbb{P}}$-supermartingale under $\mathbb{P}$. Since $f^{\mathbb{P}}(0,0,0)=0$, we can apply the downcrossing inequality proved in [95] to obtain classically that for $\mathbb{P}-a.e.\ \omega$, the limit

$$\lim_{r\in\mathbb{Q}\cup(t,T],r\downarrow t} \widetilde{V}^{\mathbb{P}}_r(\omega)$$

exists for all t.

Finally, since $\bar{\mathcal{Y}}^{\mathbb{P}}$ is càdlàg, we obtain the desired result. □

We follow now Remark 4.9 in [102], and for a fixed $\mathbb{P}\in\mathcal{P}^{\kappa}_H$, we introduce the following RBSDEJ and with lower obstacle V^+ under $\mathbb{P}$

$$\begin{aligned} &\widetilde{Y}^{\mathbb{P}}_t = \xi - \int_t^T \widehat{F}_s(\widetilde{Y}^{\mathbb{P}}_s, \widetilde{Z}^{\mathbb{P}}_s, \widetilde{U}^{\mathbb{P}}_s, \nu)ds - \int_t^T \widetilde{Z}^{\mathbb{P}}_s dB^c_s - \int_t^T\int_E \widetilde{U}^{\mathbb{P}}_s(x)\widetilde{\mu}_{B^d}(ds,dx) + \widetilde{K}^{\mathbb{P}}_T - \widetilde{K}^{\mathbb{P}}_t \\ &\widetilde{Y}^{\mathbb{P}}_t \geqslant V^+_t,\ 0\leqslant t\leqslant T,\ \mathbb{P}-a.s. \\ &\int_0^T \left(\widetilde{Y}^{\mathbb{P}}_{s^-} - V^+_{s^-}\right) d\widetilde{K}^{\mathbb{P}}_s = 0,\ \mathbb{P}-a.s. \end{aligned}$$

Remark 5.4.3. *Existence and uniqueness of the above RBSDE under our Assumptions, with the restrictions that the compensator is not random, have been proved by Hamadène and Ouknine [52] or Essaky [39]. However, their proofs can be easily generalized to our context.*

Let us now argue by contradiction and suppose that $\widetilde{Y}^{\mathbb{P}}$ is not equal $\mathbb{P}-a.s.$ to V^+. Then we can assume without loss of generality that $\widetilde{Y}_0^{\mathbb{P}} > V_0^+$, $\mathbb{P}-a.s.$ fix now some $\varepsilon > 0$ and define the following stopping-time

$$\tau^\varepsilon := \inf\left\{t \geqslant 0,\ \widetilde{Y}_t^{\mathbb{P}} \leqslant V_t^+ + \varepsilon\right\}.$$

Then $\widetilde{Y}^{\mathbb{P}}$ is strictly above the obstacle before τ^ε, and therefore $\widetilde{K}^{\mathbb{P}}$ is identically equal to 0 in $[0,\tau^\varepsilon]$. Hence, we have

$$\widetilde{Y}_t^{\mathbb{P}} = \widetilde{Y}_{\tau^\varepsilon}^{\mathbb{P}} - \int_t^{\tau^\varepsilon} \widehat{F}_s(\widetilde{Y}_s^{\mathbb{P}}, \widetilde{Z}_s^{\mathbb{P}}, \widetilde{U}_s^{\mathbb{P}})ds - \int_t^{\tau^\varepsilon} \widetilde{Z}_s^{\mathbb{P}} dB_s^c - \int_t^{\tau^\varepsilon}\int_E \widetilde{U}_s^{\mathbb{P}}(x)\widetilde{\mu}_{B^d}(ds,dx),\ \mathbb{P}-a.s.$$

Let us now define the following BSDEJ on $[0,\tau^\varepsilon]$

$$y_t^{+,\mathbb{P}} = V_{\tau^\varepsilon}^+ - \int_t^{\tau^\varepsilon} \widehat{F}_s(y_s^{+,\mathbb{P}}, z_s^{+,\mathbb{P}}, u_s^{+,\mathbb{P}})ds - \int_t^{\tau^\varepsilon} z_s^{+,\mathbb{P}} dB_s^c - \int_t^{\tau^\varepsilon}\int_E u_s^{+,\mathbb{P}}(x)\widetilde{\mu}_{B^d}(ds,dx),\ \mathbb{P}-a.s.$$

By the standard *a priori* estimates already used in this chapter, we obtain that

$$\widetilde{Y}_0^{\mathbb{P}} \leqslant y_0^{+,\mathbb{P}} + C\left|V_{\tau^\varepsilon}^+ - \widetilde{Y}_{\tau^\varepsilon}^{\mathbb{P}}\right| \leqslant y_0^{+,\mathbb{P}} + C\varepsilon,$$

by definition of τ^ε.

Following the arguments in Step 1 of the proof of Theorem 4.5 in [102], we can show that $y_0^{+,\mathbb{P}} \leqslant V_0^+$ which in turn implies

$$\widetilde{Y}_0^{\mathbb{P}} \leqslant V_0^+ + C\varepsilon,$$

hence a contradiction by arbitrariness of ε.

Therefore, we have obtained the following decomposition

$$V_t^+ = \xi - \int_t^T \widehat{F}_s(V_s^+, \widetilde{Z}_s^{\mathbb{P}}, \widetilde{U}_s^{\mathbb{P}})ds - \int_t^T \widetilde{Z}_s^{\mathbb{P}} dB_s^c - \int_t^T\int_E \widetilde{U}_s^{\mathbb{P}}(x)\widetilde{\mu}_{B^d}(ds,dx) + \widetilde{K}_T^{\mathbb{P}} - \widetilde{K}_t^{\mathbb{P}},\ \mathbb{P}-a.s.$$

Finally, we can use the result of Nutz [86] to aggregate the families $\left\{\widetilde{Z}^{\mathbb{P}},\ \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ and $\left\{\widetilde{U}^{\mathbb{P}},\ \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ into universal processes $\widetilde{Z}$ and $\widetilde{U}$.

We next prove the representation (5.3.1) for V and V^+, and that, as shown in Proposition 4.11 of [102], we actually have $V = V^+$, $\mathcal{P}_H^\kappa-q.s.$, which shows that in the case of a terminal condition in $UC_b(\Omega)$, the solution of the 2BSDEJ is actually $\mathbb{F}$-progressively measurable.

Proposition 5.4.3. *Assume that $\xi \in UC_b(\Omega)$ and that Assumptions 5.2.1 and 5.2.2 hold. Then we have*

$$V_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi) \text{ and } V_t^+ = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi),\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Besides, we also have for all t

$$V_t = V_t^+,\ \mathcal{P}_H^\kappa - q.s.$$

Proof. The proof for the representations is the same as the proof of proposition 4.10 in [102], since we also have a stability result for BSDEJs under our assumptions. For the equality between V and V^+, we also refer to the proof of Proposition 4.11 in [102]. □

Therefore, in the sequel we will use V instead of V^+.

Finally, we have to check that the minimum condition (5.2.13) holds. Fix $\mathbb{P}$ in $\mathcal{P}_H^\kappa$ and $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})$. Then, proceeding exactly as in Step 2 of the proof of Theorem 5.3.1, but introducing the process γ' of Assumption 5.2.1(iv) instead of γ, we can similarly obtain

$$V_t - y_t^{\mathbb{P}'} \geqslant \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_s' d\widetilde{K}_s^{\mathbb{P}'}\right] \geqslant \mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t \leqslant s \leqslant T} M_s'\left(\widetilde{K}_T^{\mathbb{P}'} - \widetilde{K}_t^{\mathbb{P}'}\right)\right],$$

where M' is defined as M but with γ' instead of γ.

Now let us prove that for any $n > 1$

$$\mathbb{E}_t^{\mathbb{P}'}\left[\left(\inf_{t \leqslant s \leqslant T} M_s'\right)^{-n}\right] < +\infty,\ \mathbb{P}' - a.s. \tag{5.4.13}$$

First we have

$$M_s' = \exp\left(\int_t^s \lambda_r dr + \int_t^s \eta_r \widehat{a}_r^{-1/2} dB_s^c - \frac{1}{2}\int_t^s |\eta_r|^2 dr - \int_t^s \int_E \gamma_r'(x)\widetilde{\mu}_{B^d}(dr,dx)\right)$$
$$\times \prod_{t \leqslant r \leqslant s} (1 - \gamma_r'(\Delta B_r)) e^{\gamma_r'(\Delta B_r)}.$$

Define, then $V_s = \mathcal{E}\left(\int_t^s \eta_r \widehat{a}_r^{-1/2} dB_s^c\right)$ and $W_s = \mathcal{E}\left(\int_t^s \int_E \gamma_r'(x)\widetilde{\mu}_{B^d}(dr,dx)\right)$. Notice that both these processes are strictly positive martingales, since η and γ' are bounded and we have assumed that $-\gamma'$ is strictly greater than -1. We have

$$M_s' = \exp\left(\int_t^s \lambda_r dr\right) V_s W_s.$$

Since the process λ is bounded, we have

$$\left(\inf_{t \leqslant s \leqslant T} M_s'\right)^{-n} \leqslant C\left(\inf_{t \leqslant s \leqslant T} V_s W_s\right)^{-n}$$
$$= C\left(\sup_{t \leqslant s \leqslant T} (V_s W_s)^{-1}\right)^n.$$

Using the Doob inequality for the submartingale $(V_sW_s)^{-1}$, we obtain

$$\mathbb{E}_t^{\mathbb{P}'}\left[\left(\inf_{t\leqslant s\leqslant T}M'_s\right)^{-n}\right]\leqslant C\mathbb{E}_t^{\mathbb{P}'}\left[(W_TV_T)^{-n}\right]$$
$$\leqslant C\left(\mathbb{E}_t^{\mathbb{P}'}\left[(W_T)^{-2n}\right]\mathbb{E}_t^{\mathbb{P}'}\left[(V_T)^{-2n}\right]\right)^{1/2}<+\infty,$$

where we used the fact that since η is bounded, the continuous stochastic exponential V has negative moments of any order, and where the same result holds for the purely discontinuous stochastic exponential W by Lemma 5.6.6.

Then, we have for any $p>1$

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}'}\left[\widetilde{K}_T^{\mathbb{P}'}-\widetilde{K}_t^{\mathbb{P}'}\right]\\
&=\mathbb{E}_t^{\mathbb{P}'}\left[\left(\inf_{t\leqslant s\leqslant T}M'_s\right)^{1/p}\left(\widetilde{K}_T^{\mathbb{P}'}-\widetilde{K}_t^{\mathbb{P}'}\right)\left(\inf_{t\leqslant s\leqslant T}M'_s\right)^{-1/p}\right]\\
&\leqslant\left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leqslant s\leqslant T}M'_s\left(\widetilde{K}_T^{\mathbb{P}'}-\widetilde{K}_t^{\mathbb{P}'}\right)\right]\right)^{1/p}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leqslant s\leqslant T}M_s^{'-\frac{2}{p-1}}\right]\mathbb{E}_t^{\mathbb{P}'}\left[\left(\widetilde{K}_T^{\mathbb{P}'}-\widetilde{K}_t^{\mathbb{P}'}\right)^2\right]\right)^{\frac{p-1}{2p}}\\
&\leqslant C\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[\left(\widetilde{K}_T^{\mathbb{P}'}-\widetilde{K}_t^{\mathbb{P}'}\right)^2\right]\right)^{\frac{p-1}{2p}}\left(V_t-y_t^{\mathbb{P}'}\right)^{1/p},
\end{aligned}$$

where we used (5.4.13).

Arguing as in Step (iii) of the proof of Theorem 5.3.1, the above inequality along with Proposition 5.4.3 shows that we have

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[\widetilde{K}_T^{\mathbb{P}'}-\widetilde{K}_t^{\mathbb{P}'}\right]=0,$$

that is to say that the minimum condition 5.2.13 is satisfied. This implies that the family $\left\{\widetilde{K}^{\mathbb{P}}\right\}_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ satisfies the consistency condition (i) of Theorem 5.2.1 and therefore can be aggregated by this Theorem 5.2.1.

5.4.3 Main result

We are now in position to state the main result of this section

Theorem 5.4.1. *Let $\xi\in\mathcal{L}_H^{2,\kappa}$. Under Assumptions 5.2.1 and 5.2.2, there exists a unique solution $(Y,Z,U)\in\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}\times\mathbb{J}_H^{2,\kappa}$ of the* 2BSDEJ (5.2.11).

Proof. The proof follow the lines of the proof of Theorem 4.7 in [101]. In general for a terminal condition $\xi\in\mathcal{L}_H^{2,\kappa}$, there exists by definition a sequence $(\xi_n)_{n\geqslant 0}\subset\mathrm{UC_b}(\Omega)$ such that

$$\lim_{n\to+\infty}\|\xi_n-\xi\|_{\mathbb{L}_H^{2,\kappa}}=0\text{ and }\sup_{n\geqslant 0}\|\xi_n\|_{\mathbb{L}_H^{2,\kappa}}<+\infty.$$

Let (Y^n,Z^n,U^n) be the solution to the 2RBSDE (5.2.11) with terminal condition ξ_n and

$$K_t^{\mathbb{P},n}:=Y_0^n-Y_t^n+\int_0^t\widehat{F}_s(Y_s^n,Z_s^n,U_s^n)ds+\int_0^t Z_s^n dB_s^c+\int_0^t\int_E U_s^n(x)\widetilde{\mu}_{B^d}(ds,dx),\ \mathbb{P}-a.s.$$

By the estimates of Proposition 5.3.4, we have as $n,m\to+\infty$

$$\|Y^n-Y^m\|^2_{\mathbb{D}^{2,\kappa}_H}+\|Z^n-Z^m\|^2_{\mathbb{H}^{2,\kappa}_H}+\|U^n-U^m\|^2_{\mathbb{J}^{2,\kappa}_H}+\sup_{\mathbb{P}\in\mathcal{P}^\kappa_H}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}|K^{\mathbb{P},n}_t-K^{\mathbb{P},m}_t|\right]$$
$$\leqslant C_\kappa\|\xi_n-\xi_m\|_{\mathbb{L}^{2,\kappa}_H}\to 0.$$

Extracting a subsequence if necessary, we may assume that

$$\|Y^n-Y^m\|^2_{\mathbb{D}^{2,\kappa}_H}+\|Z^n-Z^m\|^2_{\mathbb{H}^{2,\kappa}_H}+\|U^n-U^m\|^2_{\mathbb{J}^{2,\kappa}_H}$$
$$+\sup_{\mathbb{P}\in\mathcal{P}^\kappa_H}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leqslant t\leqslant T}|K^{\mathbb{P},n}_t-K^{\mathbb{P},m}_t|\right]\leqslant\frac{1}{2^n}.\tag{5.4.14}$$

This implies by Markov inequality that for all $\mathbb{P}$ and all $m\geqslant n\geqslant 0$

$$\mathbb{P}\left[\sup_{0\leqslant t\leqslant T}\left\{|Y^n_t-Y^m_t|^2+|K^{\mathbb{P},n}_t-K^{\mathbb{P},m}_t|^2\right\}+\int_0^T|\widehat{a}^{1/2}_t(Z^n_s-Z^m_s)|^2dt\right.\tag{5.4.15}$$
$$\left.+\int_0^T\int_E|U^n_t(x)-U^m_t(x)|^2\widehat{\nu}_t(dx)dt>n^{-1}\right]\leqslant Cn2^{-n}.\tag{5.4.16}$$

Define

$$Y:=\overline{\lim_{n\to+\infty}}Y^n,\ Z:=\overline{\lim_{n\to+\infty}}Z^n,\ U:=\overline{\lim_{n\to+\infty}}U^n,\ K^{\mathbb{P}}:=\overline{\lim_{n\to+\infty}}K^{\mathbb{P},n},$$

where the $\overline{\lim}$ for Z is taken componentwise and the $\overline{\lim}$ for U is taken pointwise. All those processes are clearly $\mathbb{F}^+$-progressively measurable. By (5.4.15), it follows from Borel-Cantelli Lemma that for all $\mathbb{P}$ we have $\mathbb{P}-a.s.$

$$\lim_{n\to+\infty}\left[\sup_{0\leqslant t\leqslant T}\left\{|Y^n_t-Y_t|^2+|K^{\mathbb{P},n}_t-K^{\mathbb{P}}_t|^2\right\}+\int_0^T|\widehat{a}^{1/2}_t(Z^n_s-Z_s)|^2dt\right.$$
$$\left.+\int_0^T\int_E|U^n_t(x)-U^m_t(x)|^2\widehat{\nu}_t(dx)dt\right]=0.$$

It follows that Y is càdlàg, $\mathcal{P}^\kappa_H-q.s.$, and that $K^{\mathbb{P}}$ is a càdlàg nondecreasing process, $\mathbb{P}-a.s.$ Furthermore, for all $\mathbb{P}$, sending m to infinity in (5.4.14) and applying Fatou's lemma under $\mathbb{P}$ gives us that $(Y,Z,U)\in\mathbb{D}^{2,\kappa}_H\times\mathbb{H}^{2,\kappa}_H\times\mathbb{J}^{2,\kappa}_H$.

Finally, we can proceed exactly as in the regular case ($\xi\in\mathrm{UC_b}(\Omega)$) to show that the minimum condition (5.2.13) holds. □

5.4.4 An extension of the representation formula

So far, we managed to provide wellposedness results for 2BSDEJs, by working under a set a probability measures which, if restricted to the ones for which the canonical process is a continuous local martingale is strictly smaller than the one considered in [101], Chapter 2 or Chapter 4. This is due mainly to the fact that we had to restrict ourselves to processes α of the form (5.2.3) in order to retrieve the aggregation result of Theorem 5.2.1, which was crucial to our analysis since it allowed us to define an aggregator for the family of predictable compensators.

This is clearly not very satisfying, not only from the theoretical point of view, but also from the practical one. Indeed, the set from which the processes α are allowed to be chosen corresponds in financial applications to the set of possible volatility processes for the market considered. It is therefore desirable to have the greatest generality possible. However, we emphasize that the restrictions we put on the predictable compensators ν are clearly not a problem from the point of view of the applications. Indeed, our set of compensators is strictly greater than the one associated to pure jump additive processes. Those processes, and more precisely the Lévy processes, being the most widely used in applications, our set is not really restrictive.

The aim of this section is to show that under additional assumptions, we can show that the representation formula (5.3.1) also holds for a larger set of probability measures fro which there is no longer any restrictions on the processes α. In this regard, we recall the set of probability measures $\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}}$ defined in Remark 5.2.5. We recall that every probability measure in this set satisfy the Blumenthal $0-1$ law and the martingale representation property. Moreover, exactly as in Definition 5.2.6, we define and restrict ourselves to the subset $\overline{\mathcal{P}}^{\kappa}_H$ of $\overline{\mathcal{P}}_{\widetilde{\mathcal{A}}}$. We define the following space for each $p \geqslant \kappa$,

$$\overline{\mathbb{L}}^{p,\kappa}_H := \left\{\xi,\ \|\xi\|_{\overline{\mathbb{L}}^{p,\kappa}_H} < +\infty\right\} \text{ where } \|\xi\|^p_{\overline{\mathbb{L}}^{p,\kappa}_H} := \sup_{\mathbb{P}\in\overline{\mathcal{P}}^{\kappa}_H} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leqslant t\leqslant T}{\operatorname{ess\,sup}}^{\mathbb{P}}\left(\mathbb{E}^{H,\mathbb{P}}_t[|\xi|^{\kappa}]\right)^{\frac{p}{\kappa}}\right],$$

and we let

$$\overline{\mathcal{L}}^{p,\kappa}_H := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\overline{\mathbb{L}}^{p,\kappa}_H}, \text{ for every } 1\leqslant\kappa\leqslant p.$$

We then have the following result, which is similar to Theorem 5.3 in [102]

Theorem 5.4.2. *Let $\xi \in \overline{\mathcal{L}}^{2,\kappa}_H$ and in addition to Assumptions 5.2.1 and 5.2.2, assume that*

- *F is uniformly continuous in a for $a \in D^1_{F_t}$, and for all (t,ω,y,z,u,a,ν)*

$$|F_t(\omega,y,z,u,a,\nu)| \leqslant C\left(1+\|\omega\|_t+|y|+|z|+\left|a^{1/2}\right|\right). \tag{5.4.17}$$

- *$\mathcal{P}^{\kappa}_H$ is dense in $\overline{\mathcal{P}}^{\kappa}_H$ in the sense that for any $\mathbb{P}^{\alpha,\nu}\in\overline{\mathcal{P}}^{\kappa}_H$ and for any $\varepsilon>0$, there exists $\mathbb{P}^{\alpha^{\varepsilon},\nu}\in\mathcal{P}^{\kappa}_H$ such that*

$$\mathbb{E}^{\mathbb{P}_\nu}\left[\int_0^T |(\alpha^{\varepsilon}_t)^{1/2}-\alpha^{1/2}_t|^2dt\right] \leqslant \varepsilon. \tag{5.4.18}$$

Then, we have

$$Y_0 = \sup_{\mathbb{P}\in\mathcal{P}^{\kappa}_H} y^{\mathbb{P}}_0 = \sup_{\mathbb{P}\in\overline{\mathcal{P}}^{\kappa}_H} y^{\mathbb{P}}_0,$$

where under any $\mathbb{P}:=\mathbb{P}^{\alpha,\nu}\in\overline{\mathcal{P}}^{\kappa}_H$, $(y^{\mathbb{P}},z^{\mathbb{P}},u^{\mathbb{P}})$ is the unique solution of the BSDEJ

$$y^{\mathbb{P}}_t = \xi - \int_t^T F_s(y^{\mathbb{P}}_s,z^{\mathbb{P}}_s,u^{\mathbb{P}}_s,\widehat{a}_s,\nu_s)ds - \int_t^T Z_s dB^c_s - \int_t^T\int_E u^{\mathbb{P}}_s(x)\widetilde{\mu}^{\mathbb{P}}_{B^d}(ds,dx),\ \mathbb{P}-a.s.$$

Proof. First, we remind Remark 5.3.1 ensures existence and uniqueness of the solutions of our BSDEs under any $\mathbb{P} \in \overline{\mathcal{P}}_H^\kappa$. We will proceed in two steps.

(i) $\xi \in \mathrm{UC_b}(\Omega)$

For any $\mathbb{P} := \mathbb{P}^{\alpha,\nu} \in \overline{\mathcal{P}}_H^\kappa$ and any $\varepsilon > 0$, let $\mathbb{P}^\varepsilon := \mathbb{P}^{\alpha^\varepsilon,\nu} \in \mathcal{P}_H^\kappa$ be given by (5.4.18). Using the process $L^\mathbb{P}$ defined in (5.2.7), we have $\mathbb{P} - a.s.$

$$y_t^\mathbb{P} = \xi(B_\cdot) - \int_t^T F_s(B_\cdot, y_s^\mathbb{P}, z_s^\mathbb{P}, u_s^\mathbb{P}, \widehat{a}_s, \nu_s)ds - \int_t^T \widehat{a}_s^{1/2} z_s^\mathbb{P} dL_s^{\mathbb{P},c}$$
$$- \int_t^T \int_E u_s^\mathbb{P}(x) \left(\mu_{L^{\mathbb{P},d}}(ds,dx) - \nu_s(dx)ds\right).$$

Let now $(\overline{y}^\mathbb{P}, \overline{z}^\mathbb{P}, \overline{u}^\mathbb{P})$ denote the unique solution of the following BSDEJ under $\mathbb{P}_\nu$

$$\overline{y}_t^\mathbb{P} = \xi(X_\cdot^{\alpha,\nu}) - \int_t^T F_s(X_\cdot^{\alpha,\nu}, \overline{y}_s^\mathbb{P}, \overline{z}_s^\mathbb{P}, \overline{u}_s^\mathbb{P}, \alpha_s, \nu_s)ds - \int_t^T \alpha_s^{1/2} \overline{z}_s^\mathbb{P} dB_s^c$$
$$- \int_t^T \int_E \overline{u}_s^\mathbb{P}(x) \left(\mu_{B^d}(ds,dx) - \nu_s(dx)ds\right).$$

By definition of $\mathbb{P}^{\alpha,\nu}$, we know that the distribution of $y^\mathbb{P}$ under $\mathbb{P}$ is equal to the distribution of $\overline{y}^\mathbb{P}$ under $\mathbb{P}_\nu$. Since the Blumenthal $0-1$ law also holds, this implies clearly that we have

$$y_0^\mathbb{P} = \overline{y}_0^\mathbb{P}.$$

Similarly, we define $y^{\mathbb{P}^\varepsilon}$ and $\overline{y}^{\mathbb{P}^\varepsilon}$. Then, using classical estimates from the BSDEJ theory (see [5] for instance) we have

$$\begin{aligned}
&|y_0^\mathbb{P} - y_0^{\mathbb{P}^\varepsilon}|^2 = |\overline{y}_0^\mathbb{P} - \overline{y}_0^{\mathbb{P}^\varepsilon}|^2 \\
&\leqslant C\mathbb{E}^{\mathbb{P}_\nu}\left[|\xi(X_\cdot^{\alpha,\nu}) - \xi(X_\cdot^{\alpha^\varepsilon,\nu})|^2 + \int_0^T \left|F_t(X_\cdot^\alpha, \overline{y}_t^\mathbb{P}, \overline{z}_t^\mathbb{P}, \alpha_t, \nu_t) - F_t(X_\cdot^{\alpha^\varepsilon}, \overline{y}_t^\mathbb{P}, \overline{z}_t^\mathbb{P}, \alpha_t^\varepsilon, \nu_t)\right|^2 dt\right].
\end{aligned}$$

Then, we have by (5.4.17)

$$\begin{aligned}
\left|F_t(X_\cdot^{\alpha^\varepsilon}, \overline{y}_t^\mathbb{P}, \overline{z}_t^\mathbb{P}, \alpha_t^\varepsilon, \nu_t)\right| &\leqslant C\left(1 + \left\|X^{\alpha^\varepsilon,\nu}\right\|_t + |\overline{y}_t^\mathbb{P}| + |\overline{z}_t^\mathbb{P}| + |\alpha_t^\varepsilon|^{1/2}\right) \\
&\leqslant C\left(1 + \|X^{\alpha,\nu}\| + |\overline{y}_t^\mathbb{P}| + |\overline{z}_t^\mathbb{P}| + |\alpha_t|^{1/2}\right) \\
&\quad + C\left(\left\|X^{\alpha^\varepsilon,\nu} - X^{\alpha,\nu}\right\| + |\alpha_t^\varepsilon - \alpha_t|^{\mathbf{1}/\mathbf{2}}\right). \qquad (5.4.19)
\end{aligned}$$

Using Doob's inequality and Itô's isometry, it is easy to see that (5.4.18) implies that

$$\mathbb{E}^{\mathbb{P}_\nu}\left[\sup_{0 \leqslant t \leqslant T}\left|X_t^{\alpha^\varepsilon,\nu} - X_t^{\alpha,\nu}\right|^2\right] \leqslant \varepsilon.$$

Since ξ is also uniformly continuous and bounded in ω, we can apply the dominated convergence Theorem in (5.4.19) to obtain

$$\lim_{\varepsilon \to 0}|y_0^\mathbb{P} - y_0^{\mathbb{P}^\varepsilon}| = 0.$$

This clearly implies the result in that case.

(ii) $\xi \in \overline{\mathcal{L}}_H^{2,\kappa}$

In that case, with the same notations as above, let $\xi_n \in \mathrm{UC_b}(\Omega)$ such that $\|\xi - \xi_n\|_{\mathbb{L}_H^{2,\kappa}} \underset{n\to+\infty}{\longrightarrow} 0$. Then, we define $y^{\mathbb{P},n}$ the solution of the BSDEJ with terminal condition ξ_n and generator $F_t(\cdot, \widehat{a}_s, \nu_s)$ under $\mathbb{P}$. Then, we have

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} y_0^{\mathbb{P},n} = \sup_{\mathbb{P}\in\overline{\mathcal{P}}_H^\kappa} y_0^{\mathbb{P},n}. \tag{5.4.20}$$

Moreover, using exactly the same estimates as in the proof of Theorem 5.3.4, we can show that

$$\left|y_0^{\mathbb{P},n} - y_0^{\mathbb{P}}\right|^2 \leqslant C \left\|\xi_n - \xi\right\|^2_{\mathbb{L}_H^{2,\kappa}}.$$

This shows that the convergence of $y_0^{\mathbb{P},n}$ to $y_0^{\mathbb{P}}$ is uniform with respect to $\mathbb{P} \in \overline{\mathcal{P}}_H^\kappa$. Hence we can pass to the limit in (5.4.20) and exchange the limit and the suprema to obtain the desired result. □

We finish this section by recalling a result from [102] (see Proposition 5.4) which gives a sufficient condition for the density condition (5.4.18)

Lemma 5.4.3. *Assume that the domain of F does not depend on t and that D_F^1 contains a countable dense subset. Then (5.4.18) holds.*

Proof. It suffices to notice that in our framework, all the constant mappings belong to $\widetilde{\mathcal{A}}_0$. Then Proposition 5.4 in [102] applies. □

5.5 Application to a robust utility maximization problem

In this section, we will always assume that the matrices $\underline{a} := \underline{a}^{\mathbb{P}}$ and $\overline{a} := \overline{a}^{\mathbb{P}}$ are uniformly bounded in $\mathbb{P}$. In particular, this implies that we can restrict ourselves to the case where the parameter a in the definition of a generator F is bounded. We consider a financial market consisting of one riskless asset, whose price is assumed to be equal to 1 for simplicity, and one risky asset whose price process $(S_t)_{0\leqslant t\leqslant T}$ is assumed to follow a mixed-diffusion

$$\frac{dS_t}{S_{t^-}} = b_t dt + dB_t^c + \int_E \beta_t(x)\mu_{B^d}(dt,dx),\ \overline{\mathcal{P}}_H^\kappa - q.s., \tag{5.5.1}$$

where we assume that

Assumption 5.5.1. (i) *(b_t) is a bounded $\mathbb{F}$-predictable process which is also uniformly continuous in ω.*

(ii) *(β_t) is a bounded $\mathbb{F}$-predictable process which is uniformly continuous in ω, verifies*

$$\sup_{\nu\in\mathcal{N}} \int_0^T \int_E |\beta_t(x)|\, \nu_t(dx)dt < +\infty,\ \overline{\mathcal{P}}_H^\kappa - q.s.,$$

and satisfies

$$C_2(1 \wedge |x|) \geqslant \beta_t(x) \geqslant C_1(1 \wedge |x|),\ \overline{\mathcal{P}}_H^\kappa - q.s., \text{ for all } (t,x) \in [0,T] \times E,$$

where $C_2 \geqslant 0 \geqslant C_1 > -1$.

Remark 5.5.1. *The uniform continuity assumption on* ω *is here to ensure that the 2BSDEs we will encounter in the sequel indeed have solutions. The assumption on* β *is classical and implies that the price process* S *is positive.*

Remark 5.5.2. *The volatility is implicitly embedded in the model. Indeed, under each* $\mathbb{P} \in \mathcal{P}_H^\kappa$, *we have* $dB_s^c \equiv \widehat{a}_t^{1/2} dW_t^{\mathbb{P}}$ *where* $W^{\mathbb{P}}$ *is a Brownian motion under* $\mathbb{P}$. *Therefore,* $\widehat{a}^{1/2}$ *plays the role of volatility under each* $\mathbb{P}$ *and thus allows us to model the volatility uncertainty. Similarly, we have incertitude on the jumps of our price process, since the predictable compensator of the jumps of discontinuous part of the canonical process changes with the probability considered. This allows us to have incertitude not only about the size of the jumps but also about their laws.*

We then denote $\pi = (\pi_t)_{0 \leqslant t \leqslant T}$ a trading strategy, which is a 1-dimensional $\mathcal{F}$-predictable process, supposed to take its value in some compact set C. The process π_t describes the amount of money invested in the stock at time t. The number of shares is $\frac{\pi_t}{S_{t^-}}$. So the liquidation value of a trading strategy π with positive initial capital x is given by the following wealth process:

$$X_t^\pi = x + \int_0^t \pi_s \left(dB_s^c + b_s ds + \int_E \beta_s(x) \mu_{B^d}(ds,dx) \right),\ 0 \leqslant t \leqslant T,\ \overline{\mathcal{P}}_H^\kappa - q.s.$$

The problem of the investor in this financial market is to maximize his expected exponential utility under model uncertainty from his total wealth $X_T^\pi - \xi$ where ξ is a liability at time T which is a random variable assumed to be $\mathcal{F}_T$-measurable. Then the value function V of the maximization problem can be written as

$$\begin{aligned} V^\xi(x) :&= \sup_{\pi \in \mathcal{C}} \inf_{\mathbb{P} \in \overline{\mathcal{P}}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[-\exp\left(-\eta \left(X_T^\pi - \xi \right) \right) \right] \\ &= -\inf_{\pi \in \mathcal{C}} \sup_{\mathbb{P} \in \overline{\mathcal{P}}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[\exp\left(-\eta \left(X_T^\pi - \xi \right) \right) \right]. \end{aligned} \tag{5.5.2}$$

where

$$\mathcal{C} := \{(\pi_t) \text{ which are predictable and take values in } C\},$$

is our set of admissible strategies.

Before going on, we emphasize immediately, that in the sequel we will limit ourselves to probability measures in $\mathcal{P}_H^\kappa$. We will recover the supremum over all probability measures in $\overline{\mathcal{P}}_H^\kappa$ at the end by showing that Theorem 5.4.2 applies.

To find the value function V^ξ and an optimal trading strategy π^*, we follow the ideas of the general *martingale optimality principle* approach as in [38] and [54], but adapt it here to a nonlinear framework as in Chapter chap:robust.

Let $\{R^\pi\}$ be a family of processes which satisfies the following properties

Properties 5.5.1. (i) *$R_T^\pi = \exp(-\eta(X_T^\pi - \xi))$ for all $\pi \in \mathcal{C}$.*

(ii) *$R_0^\pi = R_0$ is constant for all $\pi \in \mathcal{C}$.*

(iii) *We have*

$$R_t^\pi \leqslant \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[R_T^\pi], \ \forall \pi \in \mathcal{C}$$

$$R_t^{\pi^*} = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[R_T^{\pi^*}] \text{ for some } \pi^* \in \mathcal{C}, \ \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Then it follows

$$\sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[U(X_T^\pi - \xi)] \geqslant R_0 = \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[U(X_T^{\pi^*} - \xi)] = -V^\xi(x). \tag{5.5.3}$$

To construct R^π, we set

$$R_t^\pi = \exp(-\eta X_t^\pi) Y_t, \ t \in [0,T], \ \pi \in \mathcal{C},$$

where $(Y, Z, U) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$ the unique solution of the following 2BSDEJ

$$Y_t = e^{\eta\xi} - \int_t^T \widehat{F}_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s^c - \int_t^T \int_E U_s(x) \widetilde{\mu}_{B^d}(ds, dx) + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \ \mathcal{P}_H^\kappa - q.s. \tag{5.5.4}$$

The generator $\widehat{F}$ is chosen so that R^π satisfies the Properties 5.5.1. Let us apply Itô's formula to $\exp(-\eta X_t^\pi) Y_t$ under some $\mathbb{P} \in \mathcal{P}_H^\kappa$. We obtain after some calculations

$$\begin{aligned} d\left(e^{-\eta X_t^\pi} Y_t\right) = e^{-\eta X_{t^-}^\pi} \Bigg[& -\eta \pi_t b_t Y_t dt + \frac{\eta^2}{2} \pi_t^2 \widehat{a}_t Y_t dt - \eta \pi_t \widehat{a}_t Z_t dt + \widehat{F}_t(Y_t, Z_t, U_t) dt \\ & + \int_E \left(e^{-\eta \pi_t \beta_t(x)} - 1\right) (Y_t + U_t(x)) \widehat{\nu}_t(dx) dt + (Z_t - \eta \pi_t Y_t) dB_t^c \\ & + \int_E \left(e^{-\eta \pi_t \beta_t(x)} - 1\right) (Y_{t^-} + U_t(x)) + U_t(x) \widetilde{\mu}_{B^d}(dt, dx) - dK_t^{\mathbb{P}} \Bigg]. \end{aligned} \tag{5.5.5}$$

Hence the appropriate choice for F

$$F_s(y, z, u, a, \nu) := -\inf_{\pi \in C} \left\{ (-\eta b_s + \frac{\eta^2}{2} \pi a) \pi y - \eta \pi a z + \int_E \left(e^{-\eta \pi \beta_s(x)} - 1\right) (y + u(x)) \nu(dx) \right\}.$$

First, because of Assumption 5.5.1, F is uniformly Lipschitz in (y, z), uniformly continuous in ω. It is also continuous in a and since $D_F^1 = [\underline{a}, \overline{a}]$, it is even uniformly continuous in a. Besides, it is convex in a and ν (since it is the minus infimum of a family of linear functions) and hence can be written as a Fenchel-Legendre transform. Moreover, its domain clearly does not depend on (ω, t, y, z, u) by our boundedness assumptions. Besides, D_F^1 clearly contains a countable dense subset. This in particular shows that Theorem 5.4.2 applies here.

Finally,

$$\inf_{\pi\in C}\int_E \left(1-e^{-\eta\pi\beta_s(\omega,x)}\right)(u(x)-u'(x))\nu(dx) \leqslant F_s(\omega,y,z,u,a,\nu)-F_s(\omega,y,z,u',a,\nu)$$
$$F_s(\omega,y,z,u,a,\nu)-F_s(\omega,y,z,u',a,\nu) \leqslant \sup_{\pi\in C}\int_E \left(1-e^{-\eta\pi\beta_s(\omega,x)}\right)(u(x)-u'(x))\nu(dx).$$

Since C is compact and β is bounded, it is therefore clear from the above inequalities that Assumption 5.2.1(iv) is satisfied. Therefore, if we assume that $e^{\eta\xi}\in\mathcal{L}_H^{2,\kappa}$ (for instance if $\xi\in\mathcal{L}_H^{\infty,\kappa}$), the 2BSDEJ (5.5.4) indeed has a unique solution and R^π is well defined. Let us now prove that it satisfies the properties 5.5.1. The property (i) is clear by definition and (ii) holds because of Proposition 5.4.3. Now for any $0\leqslant t\leqslant T$, any $\pi\in\mathcal{C}$, any $\mathbb{P}\in\mathcal{P}_H^\kappa$ and any $\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})$, we have from (5.5.5)

$$\mathbb{E}_t^{\mathbb{P}'}\left[R_T^\pi\right]-R_t^\pi \geqslant -\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T e^{-\eta X_{s^-}^\pi}dK_s^{\mathbb{P}'}\right]. \tag{5.5.6}$$

Let us now prove that for any $\pi\in\mathcal{C}$ and for any $\mathbb{P}$, we have

$$\underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T e^{-\eta X_{s^-}^\pi}dK_s^{\mathbb{P}'}\right]=0. \tag{5.5.7}$$

This is similar to what we did in the proof of Theorem 5.3.1, and therefore we know that it is sufficient to prove that for any $p>1$

$$\mathbb{E}_t^{\mathbb{P}'}\left[\sup_{t\leqslant s\leqslant T} e^{-pX_s^\pi}\right]\leqslant C_p, \tag{5.5.8}$$

for some positive constant C_p depending only on p and the bounds for π, b and β.

Let M be such that $-M\leqslant\pi\leqslant M$, and let

$$N_t=e^{2M\int_0^t\int_E|\beta_s(x)|\widetilde{\mu}_s(dx)ds-\int_0^t\int_E(e^{2M|\beta_s(x)|}-1-2M|\beta_s(x)|)\widehat{\nu}_s(dx)ds},$$

then we have

$$\begin{aligned}
e^{-X_s^\pi} &\leqslant e^{\left(TM\|b\|_\infty-\int_0^t\pi_s dB_s^c+M\int_0^t\int_E|\beta_s(x)|\mu_{B^d}(ds,dx)\right)}\\
&\leqslant C\left[e^{\left(2TM\|b\|_\infty-2\int_0^t\pi_s dB_s^c\right)}+e^{\left(2M\int_0^t\int_E|\beta_s(x)|\mu_{B^d}(ds,dx)\right)}\right]\\
&= C\left[e^{\left(2TM\|b\|_\infty+2\int_0^t\pi_s^2\widehat{a}_s ds\right)}\times\mathcal{E}\left(-2\int_0^t\pi_s dB_s^c\right)\right.\\
&\qquad\left.+e^{\left(2M\int_0^t\int_E|\beta_s(x)|\widehat{\nu}_s(dx)ds+\int_0^t\int_E(e^{2M|\beta_s(x)|}-1-2M|\beta_s(x)|)\widehat{\nu}_s(dx)ds\right)}\times N_t\right]\\
&\leqslant C'\left(\mathcal{E}\left(-2\int_0^t\pi_s dB_s^c\right)+N_t\right)
\end{aligned}$$

where we used in the last inequality the fact that π and $\widehat{a}$ are uniformly bounded and that $\sup\limits_{\nu\in\mathcal{N}}\int_0^T\int_E|\beta_t(x)|\,\nu_t(dx)dt<+\infty$.

Then (5.5.8) comes from the fact that the expectations of the above Doléans-Dade exponential and N_t are finite. Using (5.5.7) in (5.5.6), we obtain

$$R_t^\pi \leqslant \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[R_T^\pi].$$

Now, using a classical measurable selection argument (see [26] (chapitre III) or [31] or Lemma 3.1 in [33]) we can define a predictable process $\pi^* \in \mathcal{C}$ such that

$$-\widehat{F}_s(Y_s, Z_s, U_s) = (-\eta b_s + \frac{\eta^2}{2}\pi_s^*\widehat{a}_s)\pi_s^* Y_s - \eta\pi_s^*\widehat{a}_s + \int_E \left(e^{-\eta\pi_s^*\beta_s(x)} - 1\right)(Y_s + U_s(x))\widehat{\nu}_s(dx).$$

Using the same arguments as above, we obtain

$$R_t^{\pi^*} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[R_T^{\pi^*}],$$

which proves (iii) of Property 5.5.1 holds.

We summarize everything in the following proposition

Proposition 5.5.1. *Assume that* $\exp(\eta\xi) \in \overline{\mathcal{L}}_H^{2,\kappa}$. *Then, under Assumption 5.5.1, the value function of the optimization problem* (5.5.2) *is given by*

$$V^\xi(x) = -e^{-\eta x} Y_0,$$

where Y_0 *is defined as the initial value of the unique solution* $(Y, Z, U) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$ *of the following 2BSDEJ*

$$Y_t = \xi - \int_t^T \widehat{F}_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s^c - \int_t^T \int_E U_s(x)\widetilde{\mu}_{B^d}(ds, dx) + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \quad (5.5.9)$$

where the generator is defined as follows

$$\widehat{F}_t(\omega, y, z, u) := F_t(\omega, y, z, u, \widehat{a}_t, \widehat{\nu}_t), \quad (5.5.10)$$

where

$$F_t(y, z, u, a, \nu) := -\inf_{\pi \in C}\left\{(-\eta b_t + \frac{\eta^2}{2}\pi a)\pi y - \eta\pi a z + \int_E \left(e^{-\eta\pi\beta_t(x)} - 1\right)(y + u(x))\nu(dx)\right\}.$$

Moreover, there exists an optimal trading strategy π^* *realizing the infimum above.*

Furthermore, by making a change of variables and applying Itô's formula, we can prove existence and uniqueness of a solution to a particular 2BSDEJ whose generator satisfies a quadratic growth condition.

Proposition 5.5.2. *Assume that $\xi \in \overline{\mathcal{L}}_H^{\infty,\kappa}$. Then, there exists a unique solution $(Y', Z', U') \in \mathbb{D}_H^{\infty,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$ to the quadratic 2BSDEJ*

$$Y'_t = \xi - \int_t^T \widehat{F}'_s(Y'_s, Z'_s, U'_s)ds - \int_t^T Z'_s dB^c_s - \int_t^T \int_E U'_s(x)\widetilde{\mu}_{B^d}(ds, dx) + K'^{\mathbb{P}}_T - K'^{\mathbb{P}}_t, \quad (5.5.11)$$

where the generator is defined as follows

$$\widehat{F}'_t(\omega, z, u) := F'_t(\omega, z, u, \widehat{a}_t, \widehat{\nu}_t), \quad (5.5.12)$$

where

$$F'_t(z, u, a, \nu) := -\inf_{\pi \in C} \left\{ \frac{\eta}{2} \left| \pi a^{1/2} - \left(a^{1/2} z + \frac{b_t + \int_E \beta_t(x)\nu(dx)}{a^{1/2}\eta} \right) \right|^2 + \frac{1}{\eta} j\left(\eta(u - \pi\beta_t)\right) \right\}$$
$$+ \left(b_t + \int_E \beta_t(x)\nu(dx) \right) z + \frac{|b_t + \int_E \beta_t(x)\nu(dx)|^2}{2a\eta},$$

where $j(u) := \int_E \left(e^{u(x)} - 1 - u(x)\right)\nu(dx)$.

Moreover, $Y'_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \overline{\mathcal{P}}_H^{\kappa}(t^+, \mathbb{P})} y'^{\mathbb{P}}_t$, where $y'^{\mathbb{P}}$ is the solution to the quadratic BSDE with the same terminal condition ξ and generator $\widehat{F}'$.

As for quadratic BSDEs and 2BSDEs, we always have a deep link between the Z-part of a solution and the BMO spaces in the case with jumps. So we need to introduce the following spaces.

$\mathbb{J}^{2,\kappa}_{\mathbb{B}MO(\mathcal{P}_H^\kappa)}$ denotes the space of predictable and $\mathcal{E}$-measurable applications $U : \Omega \times [0, T] \times E$ such that

$$\|U\|^2_{\mathbb{J}^{2,\kappa}_{\mathbb{B}MO(\mathcal{P}_H^\kappa)}} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \left\| \int_0^{\cdot} \int_E U_s(x)\widetilde{\mu}_{B^d}(ds, dx) \right\|_{\mathrm{BMO}(\mathbb{P})} < +\infty,$$

where $\|\cdot\|_{\mathrm{BMO}(\mathbb{P})}$ is the usual BMO($\mathbb{P}$) norm under $\mathbb{P}$.

$\mathbb{H}^{2,\kappa}_{\mathbb{B}MO(\mathcal{P}_H^\kappa)}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{H^{2,\kappa}_{\mathbb{B}\mathrm{MO}(\mathcal{P}^\kappa_{\mathrm{H}})}} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \left\| \int_0^{\cdot} Z_s dB^c_s \right\|_{\mathrm{BMO}(\mathbb{P})} < +\infty.$$

Now we are in position to prove the proposition.

Proof. As for the previous proposition, it is sufficient to consider the set of probability measures $\mathcal{P}_H^\kappa$.

Step 1: We can make the following change of variables: $Y'_t = \frac{1}{\eta}\log(Y_t)$, $Z'_t = \frac{1}{\eta}\frac{Z_t}{Y_t}$, $U'_t = \frac{1}{\eta}\log\left(1 + \frac{U_t}{Y_{t-}}\right)$. Then by Itô's formula and the fact that K has only predictable jumps, we can verify that the triple (Y', Z', U') satisfies (5.5.11) with $K'^{\mathbb{P}}_t = \int_0^t \frac{1}{\eta Y_s} dK^{\mathbb{P},c}_s - \sum_{0<s\leqslant t} \frac{1}{\eta}\log\left(1 - \frac{\Delta K^{\mathbb{P},d}_s}{Y_{s-}}\right)$. In particular, $K'^{\mathbb{P}}$ is nondecreasing with $K'^{\mathbb{P}}_0 = 0$.

Step 2: As in Morlais [83], we can verify that the generator $\widehat{F}'$ satisfies the following conditions.

(i) $\widehat{F}'$ has the quadratic growth property. There exists $(\alpha,\delta)\in\mathbb{R}_+\times\mathbb{R}_+^*$ such that for all (t,z,u), $\mathcal{P}_H^\kappa - q.s.$ $\left|\widehat{F}_t'(0,0)\right| \leqslant \alpha$ and

$$-\left|\widehat{F}_t'(0,0)\right| - \frac{\delta}{2}\left|\widehat{a}_t^{1/2}z\right|^2 - \frac{1}{\delta}\widehat{j}(-\eta u) \leqslant \widehat{F}_t'(z,u) \leqslant \left|\widehat{F}_t'(0,0)\right| + \frac{\delta}{2}\left|\widehat{a}_t^{1/2}z\right|^2 + \frac{1}{\delta}\widehat{j}(\gamma u),$$

where $\widehat{j}(u) := \int_E \left(e^{u(x)} - 1 - u(x)\right)\widehat{\nu}(dx)$.

(ii) We have the "local Lipschitz" condition in z, $\exists\mu>0$ and a progressively measurable process $\phi\in\mathbb{H}^{2,\kappa}_{\mathbb{B}MO(\mathcal{P}_H^\kappa)}$ such that for all (t,z,z',u), $\mathcal{P}_H^\kappa - q.s.$

$$\left|\widehat{F}_t'(z,u) - \widehat{F}_t'(z',u) - \phi_t.(\widehat{a}_t^{1/2}z - \widehat{a}_t^{1/2}z')\right| \leqslant \mu\left|\widehat{a}_t^{1/2}z - \widehat{a}_t^{1/2}z'\right|\left(\left|\widehat{a}_t^{1/2}z\right| + \left|\widehat{a}_t^{1/2}z'\right|\right).$$

(iii) For every (z,u,u') there exist two predictable and $\mathcal{E}$-measurable processes (γ_t) and (γ_t') such that

$$\widehat{F}_t'(z,u) - \widehat{F}_t'(z,u') \leqslant \int_0^t\int_E \gamma_s'(x)\left(u(x)-u'(x)\right)\widehat{\nu}(dx)ds,$$

$$\int_0^t\int_E \gamma_s(x)\left(u(x)-u'(x)\right)\widehat{\nu}(dx)ds \leqslant \widehat{F}_t'(z,u) - \widehat{F}_t'(z,u')\ \mathcal{P}_H^\kappa - q.s.,$$

where there exists constants $C_1, C_1' < 0$ and $1 > C_2, C_2' > 0$, independent of (z,u,u') such that

$$C_1(1\wedge|x|) \leqslant \gamma_t(x) \leqslant C_2(1\wedge|x|),$$

$$C_1'(1\wedge|x|) \leqslant \gamma_t'(x) \leqslant C_2'(1\wedge|x|).$$

In particular, γ and γ' are in $\mathbb{J}^{2,\kappa}_{\mathbb{B}MO(\mathcal{P}_H^\kappa)}$.

Then we know, from [83], that under each $\mathbb{P}$ the BSDEJ with the same terminal condition ξ and generator $\widehat{F}'$ has a unique solution, which we note by $(y'^{\mathbb{P}}, z'^{\mathbb{P}}, u'^{\mathbb{P}})$. Due to the monotonicity of the function $\frac{1}{\eta}\log(x)$, we have the following representation for Y': $Y_t' = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_t'^{\mathbb{P}}$.

Step 3: Next, we will prove the minimum condition for $K'^{\mathbb{P}}$. As in Chapter 2 for 2BSDEs with quadratic growth generators, we use the above representation of Y' and the conditions of $\widehat{F}'$ in z and u.

Fix $\mathbb{P}$ in $\mathcal{P}_H^\kappa$ and $\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})$, denote

$$\delta Y' := Y' - y'^{\mathbb{P}'},\ \delta Z' := Z' - z'^{\mathbb{P}'} \text{ and } \delta U' := U' - u'^{\mathbb{P}'}.$$

By the "local Lipschitz" condition (ii) of $\widehat{F}'$ in z, there exist a process η with

$$|\eta_t| \leqslant \mu\left(\left|\widehat{a}_t^{1/2}Z_t'\right| + \left|\widehat{a}_t^{1/2}z'^{\mathbb{P}'}_t\right|\right),\ \mathbb{P}' - a.s.$$

such that

$$\delta Y'_t = \int_t^T \left((\eta_s + \phi_s)\widehat{a}_s^{1/2}\delta Z'_s\right) ds - \int_t^T \delta Z'_s dB^c_s$$
$$- \int_t^T \int_E \delta U'_s(x)\left[\widetilde{\mu}_{B^d}(ds,dx) + \gamma'_s(x)\widehat{\nu}(dx)ds\right]$$
$$- \int_t^T \left[\widehat{F}'_s(z'_s, U'_s) - \widehat{F}'_s(z'_s, u'_s)\right] ds + \int_t^T \int_E \gamma'_s(x)\delta U'_s(x)\widehat{\nu}(dx)ds$$
$$+ K'^{\mathbb{P}'}_T - K'^{\mathbb{P}'}_t, \ t \leqslant T, \ \mathbb{P}' - a.s..$$

As in the proof of Lemma 2.3.1 in Chapter 2, by applying Itô's formula to $e^{-\nu Y'_t}$ for some $\nu > 0$, we have $Z \in \mathbb{H}^{2,\kappa}_{\mathbb{B}MO(\mathcal{P}^\kappa_H)}$. Then the process η defined above is also in $\mathbb{H}^{2,\kappa}_{\mathbb{B}MO(\mathcal{P}^\kappa_H)}$. So, with Girsanov's theorem we can find an equivalent probability measure $\mathbb{Q}'$ such that

$$\frac{d\mathbb{Q}'}{d\mathbb{P}'} = \mathcal{E}\left(\int_0^{\cdot} (\eta_s + \phi_s)\,\widehat{a}_s^{-1/2} dB^c_s - \int_0^{\cdot}\int_E \gamma'_s(x)\widetilde{\mu}_{B^d}(ds,dx)\right).$$

Thus, we obtain

$$Y'_t - y'^{\mathbb{P}'}_t \geqslant \mathbb{E}^{\mathbb{Q}'}_t\left[K'^{\mathbb{P}'}_T - K'^{\mathbb{P}'}_t\right].$$

For notational convenience, denote $\mathcal{E}^1_t := \mathcal{E}\left(\int_0^t (\phi_s + \eta_s)\widehat{a}_s^{-1/2} dB^c_s\right)$ and $\mathcal{E}^2_t := \mathcal{E}\left(-\int_0^t \int_E \gamma'_s(x)\widetilde{\mu}_{B^d}(ds,dx)\right)$. Let r be the number given by Lemma 2.2.2 in 2 applied to $\mathcal{E}^1$. Then we estimate

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}'}_t\left[K'^{\mathbb{P}'}_T - K'^{\mathbb{P}'}_t\right] \\
&\leqslant \mathbb{E}^{\mathbb{P}'}_t\left[\frac{\mathcal{E}_T}{\mathcal{E}_t}(K'^{\mathbb{P}'}_T - K'^{\mathbb{P}'}_t)\right]^{\frac{1}{2r-1}} \mathbb{E}^{\mathbb{P}'}_t\left[\left(\frac{\mathcal{E}_t}{\mathcal{E}_T}\right)^{\frac{1}{2(r-1)}}(K'^{\mathbb{P}'}_T - K'^{\mathbb{P}'}_t)\right]^{\frac{2(r-1)}{2r-1}} \\
&\leqslant (\delta Y'_t)^{\frac{1}{2r-1}}\left(\mathbb{E}^{\mathbb{P}'}_t\left[\left(\frac{\mathcal{E}^1_t}{\mathcal{E}^1_T}\right)^{\frac{1}{r-1}}\right]\right)^{\frac{r-1}{2r-1}}\left(\mathbb{E}^{\mathbb{P}'}_t\left[\left(\frac{\mathcal{E}^2_t}{\mathcal{E}^2_T}\right)^{\frac{2}{r-1}}\right]\mathbb{E}^{\mathbb{P}'}_t\left[(K'^{\mathbb{P}'}_T - K'^{\mathbb{P}'}_t)^4\right]\right)^{\frac{r-1}{2(2r-1)}} \\
&\leqslant C\left(\mathbb{E}^{\mathbb{P}'}_t\left[\left(K'^{\mathbb{P}'}_T\right)^4\right]\right)^{\frac{r-1}{2(2r-1)}}(\delta Y'_t)^{\frac{1}{2r-1}}.
\end{aligned}$$

With the same argument as in Step (iii) of the proof of Theorem 5.3.1, the above inequality along with the representation for Y' shows that we have

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}^\kappa_H(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[K'^{\mathbb{P}'}_T - K'^{\mathbb{P}'}_t\right] = 0,$$

that is to say that the minimum condition 5.2.13 is verified.

Step 4: Finally, by uniqueness of the solution of 2BSDEJ (5.5.9), the quadratic 2BSDEJ (5.5.11) has a unique solution. In fact, after making the reverse change of variables: $Y_t = \exp(\eta Y'_t)$, $Z_t = \exp(\eta Y'_t)\eta Z'_t$, $U_t = \exp(\eta(Y'_{t-} + U'_t)) - \exp(\eta Y'_{t-})$, we can verify that (Y, Z, U) is the solution of 2BSDE (5.5.9) with $K^{\mathbb{P}}_t = \int_0^t \exp(\eta Y'_s) dK'^{\mathbb{P},c}_s + \sum_{0<s\leqslant t}(\exp(\eta Y'_{s-}) - \exp(\eta(Y'_{s-} - \Delta K'^{\mathbb{P},d}_s)))$, where the minimum condition of $K^{\mathbb{P}}$ can be verified similarly as in Step 3. □

5.6 Appendix

5.6.1 The measures $\mathbb{P}^{\alpha,\nu}$

Lemma 5.6.1. *Let $\mathbb{P} \in \mathcal{P}_{\tilde{A}}$ and τ be an $\mathcal{F}^B$-stopping time. Then*

$$\mathbb{P}^{\tau,\omega} \in \mathcal{P}_{\tilde{A}}^{\tau(\omega)}$$

Proof.

<u>Step 1</u>: Let us first prove that

$$(\mathbb{P}_\nu)^{\tau,\omega} = \mathbb{P}^{\tau(\omega)}_{\nu^{\tau,\omega}}, \quad \mathbb{P}_\nu\text{-a.s. on } \Omega, \tag{5.6.1}$$

where $(\mathbb{P}_\nu)^{\tau,\omega}$ denotes the probability measure on Ω^τ, constructed from the r.c.p.d. of $\mathbb{P}_\nu$ for the stopping time τ, evaluated at ω, and $\mathbb{P}^{\tau(\omega)}_{\nu^{\tau,\omega}}$ is the unique solution of the martingale problem $(\mathbb{P}^1, \tau(\omega), T, Id, \nu^{\tau,\omega})$, where $\mathbb{P}^1$ is such that $\mathbb{P}^1(B^\tau_\tau = 0) = 1$.

It is enough to show that the shifted processes M^τ, J^τ, Q^τ are $(\mathbb{P}_\nu)^{\tau,\omega}$-local martingales, where M, J and Q are defined in Remark 5.2.1. For this, take a bounded $\mathcal{F}^\tau$-stopping time S. Observe that it is then clear that there exists a bounded $\mathbb{F}$-stopping time $\tilde{S}$ such that $S = \tilde{S}^{\tau,\omega}$. Then, following the definitions in Subsection 5.4.1,

$$\begin{aligned}\Delta B^{\tau,\omega}_S(\widetilde{\omega}) &= \Delta B_S(\omega \otimes_\tau \widetilde{\omega}) = \Delta(\omega \otimes_\tau \widetilde{\omega})(S)\\ &= \Delta\omega_S \mathbf{1}_{\{S \leqslant \tau\}} + \Delta\widetilde{\omega}_S \mathbf{1}_{\{S > \tau\}},\end{aligned}$$

and that for $S \geqslant \tau$

$$\begin{aligned}B_S(\omega \otimes_\tau \widetilde{\omega}) &= (\omega \otimes_\tau \widetilde{\omega})(S) = \omega_\tau + \widetilde{\omega}_S\\ &= B_\tau(\omega) + B^\tau_S(\widetilde{\omega}).\end{aligned}$$

From this we get

$$\begin{aligned}M^{\tau,\omega}_S(\widetilde{\omega}) =& M_S(\omega \otimes_\tau \widetilde{\omega})\\ =& B_S(\omega \otimes_\tau \widetilde{\omega}) - \sum_{u \leqslant S} \mathbf{1}_{|\Delta B_u(\omega\otimes_\tau\widetilde{\omega})|>1} \Delta B_u(\omega \otimes_\tau \widetilde{\omega})\\ &+ \int_0^S x\mathbf{1}_{|x|>1}\nu_u(\omega \otimes_\tau \widetilde{\omega})(dx)\,du\\ =& B^\tau_S(\widetilde{\omega}) + B_t(\omega) - \sum_{u \leqslant \tau} \mathbf{1}_{|\Delta\omega_u|>1}\Delta\omega_u - \sum_{\tau<u \leqslant S} \mathbf{1}_{|\Delta B^\tau_u(\widetilde{\omega})|>1}\Delta B^\tau_u(\widetilde{\omega})\\ &+ \int_0^\tau x\mathbf{1}_{|x|>1}\nu_u(\omega)(dx)\,du + \int_\tau^S x\mathbf{1}_{|x|>1}\nu^{\tau,\omega}_u(\widetilde{\omega})(dx)\,du\\ =& M^\tau_S(\widetilde{\omega}) + M_\tau(\omega),\end{aligned}$$

and we can now compute

$$\begin{aligned}\mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[M^\tau_S\right] &= \mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[M^{\tau,\omega}_S - M_\tau(\omega)\right]\\ &= \mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[M^{\tau,\omega}_{\tilde{S}^{\tau,\omega}}\right] - M_\tau(\omega)\\ &= \mathbb{E}^{\mathbb{P}_\nu}_\tau[M_{\tilde{S}}](\omega) - M_\tau(\omega) = 0, \text{for } \mathbb{P}_\nu\text{-a.e. } \omega.\end{aligned}$$

Since S is an arbitrary bounded stopping time, we have that M^τ is a $(\mathbb{P}_\nu)^{\tau,\omega}$-local martingale for $\mathbb{P}_\nu$-a.e. ω..

We treat the case of the process J^τ analogously and write

$$\begin{aligned}
J_S^{\tau,\omega}(\widetilde{\omega}) =& (M_S^{\tau,\omega}(\widetilde{\omega}))^2 - S - \int_0^\tau \int_E x^2 \nu_u(\omega)(dx)du - \int_\tau^S \int_E x^2 \nu_u^{\tau,\omega}(\widetilde{\omega})(dx)du \\
=& (M_S^\tau(\widetilde{\omega}))^2 + (M_\tau(\omega))^2 + 2M_S^\tau(\widetilde{\omega})M_\tau(\omega) - (S-\tau) - \int_\tau^S \int_E x^2 \nu_u^{\tau,\omega}(\widetilde{\omega})(dx)du \\
& - \int_0^\tau \int_E x^2 \nu_u(\omega)(dx)du - \tau \\
=& J_S^\tau(\widetilde{\omega}) + J_\tau(\omega) + 2M_S^\tau(\widetilde{\omega})M_\tau(\omega).
\end{aligned}$$

Then we can compute the expectation

$$\mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[J_S^\tau\right] = \mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[J_S^{\tau,\omega} - 2M_S^\tau M_\tau(\omega)\right] - J_\tau(\omega) = 0, \text{for } \mathbb{P}_\nu\text{-a.e. } \omega.$$

J^τ is then a $(\mathbb{P}_\nu)^{\tau,\omega}$-local martingale for $\mathbb{P}_\nu$-a.e. ω.. Finally, we do the same kind of calculation for Q^τ, and we obtain

$$\begin{aligned}
Q_S^{\tau,\omega}(\widetilde{\omega}) =& \int_0^S \int_E g(x)\mu_B(\omega \otimes_\tau \widetilde{\omega}, dx, du) - \int_0^S \int_E g(x)\nu_u^{\tau,\omega}(\widetilde{\omega})(dx)\, du \\
=& \int_0^\tau \int_E g(x)\mu_B(\omega, dx, du) + \int_\tau^S \int_E g(x)\mu_{B^\tau}(\widetilde{\omega}, dx, du) \\
& - \int_0^\tau \int_E g(x)\nu_u(\omega)(dx)du - \int_\tau^S \int_E g(x)\nu_u^{\tau,\omega}(\widetilde{\omega})(dx)\, du \\
=& Q_S^\tau(\widetilde{\omega}) + Q_\tau(\omega).
\end{aligned}$$

And again we compute the expectation over the $\widetilde{\omega} \in \Omega^\tau$, under the measure $(\mathbb{P}_\nu)^{\tau,\omega}$

$$\begin{aligned}
\mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[Q_S^\tau\right] &= \mathbb{E}^{(\mathbb{P}_\nu)^{\tau,\omega}}\left[Q_S^{\tau,\omega} - Q_\tau(\omega)\right] \\
&= \mathbb{E}_\tau^{\mathbb{P}_\nu}[Q_S](\omega) - Q_\tau(\omega) = 0, \text{for } \mathbb{P}_\nu\text{-a.e. } \omega.
\end{aligned}$$

We have the desired result, and conclude that (5.6.1) holds true.

We can now deduce that for any $(\alpha,\nu) \in \tilde{\mathcal{A}}$

$$\mathbb{P}^{\alpha^{\tau,\omega},\nu^{\tau,\omega}} \in \mathcal{P}_{\tilde{\mathcal{A}}}^{\tau(\omega)}\ \mathbb{P}_\nu\text{-a.s. on } \Omega. \tag{5.6.2}$$

Indeed, if $(\alpha,\nu) \in \mathcal{D} \times \mathcal{N}$, then $(\alpha^{\tau,\omega},\nu^{\tau,\omega}) \in \mathcal{D}^{\tau(\omega)} \times \mathcal{N}^{\tau(\omega)}$, because

$$\begin{aligned}
&\int_{\tau(\omega)}^T \int_E (1 \wedge |x|^2)\nu_s^{\tau,\omega}(\widetilde{\omega})(dx)ds < +\infty, \quad \int_{\tau(\omega)}^T \int_{|x|>1} |x|\, \nu_s^{\tau,\omega}(\widetilde{\omega})(dx)ds < +\infty, \\
&\int_{\tau(\omega)}^T |\alpha_s^{\tau,\omega}(\widetilde{\omega})|\, ds < \infty.
\end{aligned}$$

Moreover, if (α,ν) have the form (5.2.3), it is clear that it also holds true for $(\alpha^{\tau,\omega},\nu^{\tau,\omega})$.

Step 2: We define $\tilde{\tau} := \tau \circ X^\alpha$, $\tilde{\alpha}^{\tau,\omega} := \alpha^{\tilde{\tau},\beta_\alpha(\omega)}$ and $\tilde{\nu}^{\tau,\omega} := \nu^{\tilde{\tau},\beta_\alpha(\omega)}$ where β_α is a measurable map such that $B = \beta_\alpha(X^\alpha)$, $\mathbb{P}_\nu$-a.s. Moreover, $\tilde{\tau}$ is a stopping time and we have $\tau = \tilde{\tau} \circ \beta_\alpha$ since

$$\tilde{\tau} \circ \beta_\alpha = \tau \circ \beta_\alpha(X^\alpha) = \tau \circ B = \tau$$

and using (5.6.2),

$$\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}} \in \mathcal{P}_{\tilde{\mathcal{A}}}^{\tau(\omega)} \;\; \mathbb{P}^{\alpha,\nu}\text{-a.s. on } \Omega.$$

Step 3: We show that

$$\mathbb{E}^{\mathbb{P}^{\alpha,\nu}}\left[\phi\left(B_{t_1\wedge\tau},\ldots,B_{t_n\wedge\tau}\right)\psi\left(B_{t_1},\ldots,B_{t_n}\right)\right] = \mathbb{E}^{\mathbb{P}_{\alpha,\nu}}\left[\phi\left(B_{t_1\wedge\tau},\ldots,B_{t_n\wedge\tau}\right)\psi_\tau\right]$$

for every $0 < t_1 < \cdots < t_n \leqslant T$, every continuous and bounded functions ϕ and ψ and

$$\psi_\tau(\omega) = \mathbb{E}^{\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}}}\left[\psi(\omega(t_1),\ldots,\omega(t_k),\omega(t)+B^t_{t_{k+1}},\ldots,\omega(t)+B^t_{t_n})\right],$$

for $t := \tau(\omega) \in [t_k, t_{k+1}[$.

Recall that $\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}}$ is defined by $\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}} = \mathbb{P}_{\tilde{\nu}^{\tau,\omega}} \circ \left(X^{\tilde{\alpha}^{\tau,\omega}}\right)^{-1}$, then

$$\begin{aligned}\psi_\tau(\omega) = \mathbb{E}^{\mathbb{P}^{\tau(\omega)}_{\tilde{\nu}^{\tau,\omega}}}\Big[\psi\Big(&\omega(t_1),\ldots,\omega(t_k),\omega(t)+\int_t^{t_{k+1}}\left(\alpha_s^{\tilde{\tau},\beta_\alpha(\omega)}\right)^{1/2}d(B^c_s)^{\tau(\omega)}\\ &+\int_t^{t_{k+1}}\int_E x(\mu_{B^{\tau(\omega)}}(ds,dx)-\nu_s^{\tilde{\tau},\beta_\alpha(\omega)}(dx)ds),\ldots,\omega(t)+\int_t^{t_n}\left(\alpha_s^{\tilde{\tau},\beta_\alpha(\omega)}\right)^{1/2}d(B^c_s)^{\tau(\omega)}\\ &+\int_t^{t_n}\int_E x(\mu_{B^{\tau(\omega)}}(ds,dx)-\nu_s^{\tilde{\tau},\beta_\alpha(\omega)}(dx)ds)\Big)\Big].\end{aligned}$$

Then, $\forall \omega \in \Omega$, if $t := \tilde{\tau}(\omega) = \tau\left(X^\alpha(\omega)\right) \in [t_k, t_{k+1}[$,

$$\begin{aligned}\psi_\tau\left(X^\alpha(\omega)\right) = \mathbb{E}^{\mathbb{P}^{\tilde{\tau}(\omega)}_{\nu^{\tilde{\tau},\omega}}}\Big[\psi\Big(&X^\alpha_{t_1}(\omega),\ldots,X^\alpha_{t_k}(\omega),X^\alpha_t(\omega)+\int_t^{t_{k+1}}\left(\alpha_s^{\tilde{\tau},\omega}\right)^{1/2}d(B_s^{\tilde{\tau}(\omega)})^c\\ &+\int_t^{t_{k+1}}\int_E x(\mu_{B^{\tilde{\tau}(\omega)}}(ds,dx)-\nu_s^{\tilde{\tau},\omega}(dx)ds),\ldots,X^\alpha_t(\omega)+\int_t^{t_n}\left(\alpha_s^{\tilde{\tau},\omega}\right)^{1/2}d(B_s^{\tilde{\tau}(\omega)})^c\\ &+\int_t^{t_n}\int_E x(\mu_{B^{\tilde{\tau}(\omega)}}(ds,dx)-\nu_s^{\tilde{\tau},\omega}(dx)ds)\Big)\Big].\end{aligned} \tag{5.6.3}$$

We remark that for every $\omega \in \Omega$,

$$\begin{aligned}&\alpha_s(\omega) = \alpha_s\left(\omega \otimes_{\tilde{\tau}(\omega)} \omega^{\tilde{\tau}(\omega)}\right) = \alpha_s^{\tilde{\tau},\omega}\left(\omega^{\tilde{\tau}(\omega)}\right)\\ \text{and }\;&\nu_s(\omega)(dx) = \nu_s\left(\omega \otimes_{\tilde{\tau}(\omega)} \omega^{\tilde{\tau}(\omega)}\right)(dx) = \nu_s^{\tilde{\tau},\omega}\left(\omega^{\tilde{\tau}(\omega)}\right)(dx)\end{aligned}$$

By definition, the $(\mathbb{P}_\nu)^{\tilde{\tau},\omega}$-distribution of $B^{\tilde{\tau}(\omega)}$ is equal to the $(\mathbb{P}_\nu)^{\omega}_{\tilde{\tau}}$-distribution of $(B. -$

$B_{\tilde{\tau}(\omega)})$. (5.6.3) then becomes

$$\begin{aligned}
\psi_\tau\left(X^\alpha(\omega)\right) =& \mathbb{E}^{(\mathbb{P}_\nu)^\omega_{\tilde{\tau}}}\Big[\psi\Big(X^\alpha_{t_1}(\omega),\dots,X^\alpha_{t_k}(\omega),X^\alpha_t(\omega)+\int_t^{t_{k+1}}\alpha_s^{1/2}(B^c)d(B^c_s)\\
&+\int_t^{t_{k+1}}\int_E x(\mu_B(ds,dx)-\nu_s(dx)ds),\dots,X^\alpha_t(\omega)+\int_t^{t_n}\alpha_s^{1/2}(B^c)d(B^c_s)\\
&+\int_t^{t_n}\int_E x(\mu_B(ds,dx)-\nu_s(dx)ds)\Big)\Big]\\
=& \mathbb{E}^{(\mathbb{P}_\nu)^\omega_{\tilde{\tau}}}\Big[\psi\Big(X^\alpha_{t_1},\dots,X^\alpha_{t_k},X^\alpha_{t_{k+1}},\dots,X^\alpha_{t_n}\Big)\Big]\\
=& \mathbb{E}^{\mathbb{P}_\nu}\left[\psi\Big(X^\alpha_{t_1},\dots,X^\alpha_{t_k},X^\alpha_{t_{k+1}},\dots,X^\alpha_{t_n}\Big)|\mathcal{F}_{\tilde{\tau}}\right](\omega),\ \ \mathbb{P}_\nu\text{-a.s. on } \Omega.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{\alpha,\nu}}\left[\phi\left(B_{t_1\wedge\tau},\dots,B_{t_n\wedge\tau}\right)\psi_\tau\right] = \mathbb{E}^{\mathbb{P}_\nu}\left[\phi\Big(X^\alpha_{t_1\wedge\tilde{\tau}},\dots,X^\alpha_{t_n\wedge\tilde{\tau}}\Big)\psi_{\tilde{\tau}}(X^\alpha)\right]\\
&= \mathbb{E}^{\mathbb{P}_\nu}\left[\phi\Big(X^\alpha_{t_1\wedge\tilde{\tau}},\dots,X^\alpha_{t_n\wedge\tilde{\tau}}\Big)\mathbb{E}^{\mathbb{P}_\nu}\left[\psi\Big(X^\alpha_{t_1},\dots,X^\alpha_{t_k},X^\alpha_{t_{k+1}},\dots,X^\alpha_{t_n}\Big)|\mathcal{F}_{\tilde{\tau}}\right]\right]\\
&= \mathbb{E}^{\mathbb{P}_\nu}\left[\phi\Big(X^\alpha_{t_1\wedge\tilde{\tau}},\dots,X^\alpha_{t_n\wedge\tilde{\tau}}\Big)\psi\Big(X^\alpha_{t_1},\dots,X^\alpha_{t_k},X^\alpha_{t_{k+1}},\dots,X^\alpha_{t_n}\Big)\right]\\
&= \mathbb{E}^{\mathbb{P}_{\alpha,\nu}}\left[\phi\left(B_{t_1\wedge\tau},\dots,B_{t_n\wedge\tau}\right)\psi\left(B_{t_1},\dots,B_{t_n}\right)\right].
\end{aligned}$$

Step 4: Now we prove that $\mathbb{P}^{\tau,\omega}=\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}}$, $\mathbb{P}$-a.s. on Ω.

By definition of the conditional expectation,

$$\psi_\tau(\omega)=\mathbb{E}^{\mathbb{P}^{\tau,\omega}}\left[\psi(\omega(t_1),\dots,\omega(t_k),\omega(t)+B^t_{t_{k+1}},\dots,\omega(t)+B^t_{t_n})\right],\ \mathbb{P}^{\alpha,\nu}\text{-a.s.},$$

where $t:=\tau(\omega)\in[t_k,t_{k+1}[$, and where the $\mathbb{P}^{\alpha,\nu}$-null set can depend on $(t_1,\dots,t_n)$ and ψ, but we can choose a common null set by standard approximation arguments.

Then by a density argument we obtain

$$\mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\eta]=\mathbb{E}^{\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}}}[\eta],\ \text{for } \mathbb{P}^{\alpha,\nu}\text{-a.e. } \omega,$$

for every bounded and $\mathcal{F}^{\tau(\omega)}_T$-measurable random variable η. This implies $\mathbb{P}^{\tau,\omega}=\mathbb{P}^{\tilde{\alpha}^{\tau,\omega},\tilde{\nu}^{\tau,\omega}}$, $\mathbb{P}$-a.s. on Ω. And from the Step 1 we deduce that $\mathbb{P}^{\tau,\omega}\in\overline{\mathcal{P}}^{\tau(\omega)}_{\tilde{\mathcal{A}}}$. □

Lemma 5.6.2. *We have $\mathbb{P}^n\in\mathcal{P}^\kappa_H$, where $\mathbb{P}^n$ is defined by (5.4.10).*

Proof. Since by definition, $\mathbb{P}^i_t\in\mathcal{P}^t_H$ and $\mathbb{P}\in\mathcal{P}_H$, we have $\mathbb{P}^i_t=\mathbb{P}^{\alpha^i,\nu^i}$ and $\mathbb{P}=\mathbb{P}^{\alpha,\nu}$, for $(\alpha^i,\nu^i)\in\widetilde{\mathcal{A}}^t$ and $(\alpha,\nu)\in\widetilde{\mathcal{A}}$, $i=1,\dots,n$. Next we define

$$\overline{\alpha}_s:=\alpha_s\mathbf{1}_{[0,t)}(s)+\left[\sum_{i=1}^n\alpha^i_s\mathbf{1}_{E^i_t}(X^\alpha)+\alpha_s\mathbf{1}_{\hat{E}^n_t}(X^\alpha)\right]\mathbf{1}_{[t,T]}(s),\ \text{and}$$

$$\overline{\nu}_s:=\nu_s\mathbf{1}_{[0,t)}(s)+\left[\sum_{i=1}^n\nu^i_s\mathbf{1}_{E^i_t}(X^\alpha)+\nu_s\mathbf{1}_{\hat{E}^n_t}(X^\alpha)\right]\mathbf{1}_{[t,T]}(s).$$

Now following the arguments in the proof of step 3 of Lemma 5.6.1, we prove that for any $0 < t_1 < \cdots < t_k = t < t_{k+1} < t_n$ and any continuous and bounded functions ϕ and ψ,

$$\mathbb{E}^{\mathbb{P}^{\alpha,\nu}}\left[\phi(B_{t_1},\ldots,B_{t_k})\sum_{i=1}^{n}\mathbb{E}^{\mathbb{P}^{\alpha^i_t,\nu^i_t}}\left[\psi(B_{t_1},\ldots,B_{t_k},B_t+B^t_{t_{k+1}},\ldots,B_t+B^t_{t_n})\right]\mathbf{1}_{E^i_t}\right]$$
$$= \mathbb{E}^{\mathbb{P}^{\overline{\alpha},\overline{\nu}}}\left[\phi(B_{t_1},\ldots,B_{t_k})\psi(B_{t_1},\ldots,B_{t_n})\right].$$

This implies that $\mathbb{P}^n = \mathbb{P}^{\overline{\alpha},\overline{\nu}} \in \mathcal{P}_{\tilde{\mathcal{A}}}$. And since all the probability measures $\mathbb{P}^i$ satisfy the requirements of Definition 5.4.1, we have $\mathbb{P}^n = \mathbb{P}^{\overline{\alpha},\overline{\nu}} \in \mathcal{P}^\kappa_H$. $\square$

Lemma 5.6.3. *Fix an arbitrary measure $\mathbb{P} = \mathbb{P}^{\alpha,\nu}$ in $\mathcal{P}^\kappa_H$. The set $\mathcal{P}^\kappa_H(t^+,\mathbb{P})$ is upward directed, i.e. for each $\mathbb{P}_1 := \mathbb{P}^{\alpha_1,\nu_1}$ and $\mathbb{P}_2 := \mathbb{P}^{\alpha_2,\nu_2}$ in $\mathcal{P}^\kappa_H(t^+,\mathbb{P})$, there exists $\mathbb{P}' \in \mathcal{P}^\kappa_H(t^+,\mathbb{P})$ such that $\forall u > t$,*

$$\mathbb{E}^{\mathbb{P}'}_t\left[\left(K^{\mathbb{P}'}_u - K^{\mathbb{P}'}_t\right)^2\right] = \max\left\{\mathbb{E}^{\mathbb{P}_1}_t\left[\left(K^{\mathbb{P}_1}_u - K^{\mathbb{P}_1}_t\right)^2\right], \mathbb{E}^{\mathbb{P}_2}_t\left[\left(K^{\mathbb{P}_2}_u - K^{\mathbb{P}_2}_t\right)^2\right]\right\}. \tag{5.6.4}$$

Proof.

We define the following $\mathcal{F}_t$-measurable sets

$$E_1 := \left\{\omega \in \Omega \,:\, \mathbb{E}^{\mathbb{P}_2}_t\left[\left(K^{\mathbb{P}_2}_u - K^{\mathbb{P}_2}_t\right)^2\right](\omega) \leqslant \mathbb{E}^{\mathbb{P}_1}_t\left[\left(K^{\mathbb{P}_1}_u - K^{\mathbb{P}_1}_t\right)^2\right](\omega)\right\}$$

and $E_2 := \Omega\backslash E_1$. Then for all $A \in \mathcal{F}_T$, we define the probability measure $\mathbb{P}'$ by,

$$\mathbb{P}'(A) := \mathbb{P}_1(A\cap E_1) + \mathbb{P}_2(A\cap E_2).$$

By definition, $\mathbb{P}'$ satisfies (5.6.4). Let us prove now that $\mathbb{P}' \in \mathcal{P}^\kappa_H(t^+,\mathbb{P})$. As in the proof of claim (4.17) in [101], for $s \in [0,T]$, we define the processes α^* and ν^* as follows

$$\alpha^*_s(\omega) = \alpha_s(\omega)\mathbf{1}_{[0,t)}(s) + \left(\alpha^1_s(\omega)\mathbf{1}_{\{X^\alpha\in E_1\}}(\omega) + \alpha^2_s(\omega)\mathbf{1}_{\{X^\alpha\in E_2\}}(\omega)\right)\mathbf{1}_{[t,T]}(s),$$
$$\nu^*_s(\omega) = \nu_s(\omega)\mathbf{1}_{[0,t)}(s) + \left(\nu^1_s(\omega)\mathbf{1}_{\{X^\alpha\in E_1\}}(\omega) + \nu^2_s(\omega)\mathbf{1}_{\{X^\alpha\in E_2\}}(\omega)\right)\mathbf{1}_{[t,T]}(s),$$

where X^α is defined in (5.2.6). We have

$$0 < \underline{\alpha}\wedge\underline{\alpha}^1\wedge\underline{\alpha}^2 \leqslant \alpha^* \leqslant \overline{\alpha}\vee\overline{\alpha}^1\vee\overline{\alpha}^2,$$

where $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\alpha}^i$, $\overline{\alpha}^i$ are the lower and upper bounds of the processes α, α^1 and α^2. Next, we have

$$\int_0^T\int_E(1\wedge|x|^2)\nu^*_s(ds,dx) \leqslant \int_0^T\int_E(1\wedge|x|^2)\nu_s(ds,dx) + \int_0^T\int_E(1\wedge|x|^2)\nu^1_s(ds,dx)$$
$$+\int_0^T\int_E(1\wedge|x|^2)\nu^2_s(ds,dx) < +\infty,$$

and the same way we see that $\int_0^T\int_{\{|x|>1\}} x\nu^*_s(ds,dx) < +\infty$. Then, we have therefore clearly $(\alpha^*,\nu^*) \in \tilde{\mathcal{A}}$, and we can define the element $\mathbb{P}^{\alpha^*,\nu^*}$ of $P_{\tilde{\mathcal{A}}}$.

Now using the same arguments as in the Step 3 of the proof of the previous Lemma, we obtain that for any $t_1 < \cdots < t_k = t < t_{k+1} < \cdots < t_n$ and any bounded and continuous functions ϕ and ψ,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{\alpha^*,\nu^*}}\Big[\phi(B_{t_1},\dots,B_{t_k})\psi(B_{t_1},\dots,B_{t_n})\Big]\\
&= \mathbb{E}^{\mathbb{P}^{\alpha,\nu}}\Big[\phi(B_{t_1},\dots,B_{t_k})\Big(\mathbb{E}^{\mathbb{P}^{\alpha_1,\nu_1}}[\psi(B_{t_1},\dots,B_{t_k},B_t+B^t_{t_{k+1}},\dots,B_t+B^t_{t_n})]\mathbf{1}_{E_1}\\
&\quad + \mathbb{E}^{\mathbb{P}^{\alpha_2,\nu_2}}[\psi(B_{t_1},\dots,B_{t_k},B_t+B^t_{t_{k+1}},\dots,B_t+B^t_{t_n})]\mathbf{1}_{E_2}\Big)\Big].
\end{aligned}$$

This shows that

$$(\mathbb{P}^{\alpha^*,\nu^*})^\omega_t = (\mathbb{P}')^\omega_t \text{ for } \mathbb{P}^{\alpha^*,\nu^*}\text{-almost every } \omega \text{ in } \Omega \text{ and every } t > 0.$$

Thus $\mathbb{P}' = \mathbb{P}^{\alpha^*,\nu^*} \in \mathcal{P}_{\tilde{\mathcal{A}}}$. To prove that $\mathbb{P}' \in \mathcal{P}^\kappa_H$, we compute

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}'}\left[\int_0^T \left|\widehat{F}^0_s\right|^2 ds\right] &= \mathbb{E}^{\mathbb{P}}\left[\int_0^t \left|\widehat{F}^0_s\right|^2 ds\right] + \mathbb{E}^{\mathbb{P}_1}\left[\int_t^T \left|\widehat{F}^0_s\right|^2 ds\,\mathbf{1}_{E_1}\right] + \mathbb{E}^{\mathbb{P}_2}\left[\int_t^T \left|\widehat{F}^0_s\right|^2 ds\,\mathbf{1}_{E_2}\right]\\
&\leqslant \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{F}^0_s\right|^2 ds\right] + \mathbb{E}^{\mathbb{P}_1}\left[\int_0^T \left|\widehat{F}^0_s\right|^2 ds\,\mathbf{1}_{E_1}\right] + \mathbb{E}^{\mathbb{P}_2}\left[\int_0^T \left|\widehat{F}^0_s\right|^2 ds\,\mathbf{1}_{E_2}\right] < +\infty.
\end{aligned}$$

Since by construction $\mathbb{P}'$ coincides with $\mathbb{P}$ on $\mathcal{F}_t$, the proof is complete. □

5.6.2 L^r-Integrability of exponential martingales

Lemma 5.6.4. *Let $\delta > 0$ and $n \in \mathbb{N}^*$. Then there exists a constant $C_{n,\delta}$ depending only on δ and n such that*

$$(1+x)^{-n} - 1 + nx \leqslant C_{n,\delta}x^2, \text{ for all } x \in [-1+\delta, +\infty).$$

Proof. Define for $x \geqslant -1+\delta$ and for any $C > 0$ the function

$$f_C(x) := (1+x)^{-n} - 1 + nx - Cx^2.$$

First, we have

$$f_C(-1+\delta) = \delta^{-n} - 1 + n(-1+\delta) - C(-1+\delta)^2.$$

Since this quantity goes to $-\infty$ when C goes to $+\infty$, it is clear that we can choose C large enough so that $f_C(-1+\delta) \leqslant 0$.

Let us now study the function f_C. We have for any $x \geqslant -1+\delta$

$$f'_C(x) = -n(1+x)^{-n-1} + n - 2Cx = \frac{(n-2Cx)(1+x)^{n+1} - n}{(1+x)^{n+1}}.$$

Define the function

$$\begin{aligned}
g_C(x) := (n-2Cx)(1+x)^{n+1} - n &= x\left(n\frac{(1+x)^{n+1}-1}{x} - 2C(1+x)^{n+1}\right)\\
&= x\left(n\sum_{k=0}^n C^{k+1}_{n+1}x^k - 2C(1+x)^{n+1}\right).
\end{aligned}$$

Consider then

$$h_C(x) := n\sum_{k=0}^{n} C_{n+1}^{k+1}x^k - 2C(1+x)^{n+1}.$$

For all $0 \leqslant l \leqslant n$, we have by differentiating repeatedly

$$h_C^{(l)}(x) = n\sum_{k=0}^{n-l}\frac{(k+l)!}{l!}C_{n+1}^{k+l+1}x^k - 2C\frac{(n+1)!}{(n-l+1)!}(1+x)^{n-l+1}.$$

It is clear in this expression that we can always choose C large enough so that for every $l \leqslant n$, $h_C^{(l)}(-1+\delta) \leqslant 0$. Then, we have after some calculations

$$h_C^{(n-1)}(x) = n!\left(-C(n+1)x^2 + (n-2C(n+1))x + (n+1)(1-C)\right).$$

The roots of this second degree polynomial are given by

$$x = \frac{n-2C(n+1)+\sqrt{n^2+4c(n+1)}}{2C(n+1)} \text{ or } x = \frac{n-2C(n+1)-\sqrt{n^2+4c(n+1)}}{2C(n+1)}.$$

Since both these roots can be made as close as we want to -1 by choosing C large enough, we can conclude that for C large enough, we will have $h_C^{(n-1)}(x) \leqslant 0$, for $x \geqslant -1+\delta$. Hence, the function $h_C^{(n-2)}$ is decreasing for $x \geqslant -1+\delta$. But since we recalled earlier that $h_C^{(l)}(-1+\delta) \leqslant 0$, we also have $h_C^{(n-2)}(x) \leqslant 0$, for $x \geqslant -1+\delta$. Repeating those arguments, we show recursively that the function h_C itself is decreasing for $x \geqslant -1+\delta$ and since we also have $h_C(-1+\delta) \leqslant 0$, we finally obtain that the function h_C is negative for $x \geqslant -1+\delta$.

Therefore, the function g_C is positive for $x \leqslant 0$ and negative for $x \geqslant 0$. Since $f_C(-1+\delta) \leqslant 0$ and $f_C(0) = 0$, this ends the proof. □

Mémin [80] and then Lépingle and Mémin [72] proved some useful multiplicative decompositions of exponential semimartingales. We give here one of these representations that we will use in the proof of Lemma 5.6.5.

Proposition 5.6.1 (Proposition II.1 of [71])**.** *Let N be a local martingale and let A be a predictable process with finite variation such that $\Delta A \neq -1$. We assume $N_0 = A_0 = 0$. Then there exists a local martingale $\tilde{N}$ with $\tilde{N}_0 = 0$ and such that*

$$\mathcal{E}(N+A) = \mathcal{E}(\tilde{N})\mathcal{E}(A).$$

Lemma 5.6.5. *Let $\lambda > 0$ and M be a local martingale with bounded jumps, such that $\Delta M \geqslant -1+\delta$, for a fixed $\delta > 0$. Let $V^{-\lambda}$ be the predictable compensator of*

$$\left\{W_t^{-\lambda} = \sum_{s\leqslant t}\left[(1+\Delta M_s)^{-\lambda} - 1 + \lambda\Delta M_s\right],\ t \geqslant 0\right\}.$$

We have

(i) $\mathcal{E}^{-\lambda}(M) = \mathcal{E}(N^{-\lambda} + A^{-\lambda})$ *where*

$$A^{-\lambda} = \frac{\lambda(\lambda+1)}{2} \langle M^c, M^c \rangle^T + V^{-\lambda}$$
$$N^{-\lambda} = -\lambda M^T + W^{-\lambda} - V^{-\lambda}.$$

(ii) There exists a local martingale $\tilde{N}^{-\lambda}$ *such that*

$$\mathcal{E}^{-\lambda}(M) = \mathcal{E}(\tilde{N}^{-\lambda})\mathcal{E}(A^{-\lambda}).$$

Proof. First note that thanks to lemma 5.6.4, for $\lambda > 0$, $(1+x)^{-\lambda} - 1 + \lambda x \leqslant Cx^2$, and thus $W^{-\lambda}$ is integrable. We set

$$T_n = \inf\{t \geqslant 0 \ : \ \mathcal{E}(M)_t \leqslant \frac{1}{n}\} \text{ and } M_t^n = M_{t \wedge T_n}.$$

Then M^n and $\mathcal{E}(M^n)$ are local martingales, $\mathcal{E}(M^n) \geqslant \frac{1}{n}$ and $\mathcal{E}(M^n)_t = \mathcal{E}(M)_t$ if $t < T_n$. The assumption $\Delta M > -1$ shows that T_n tends to infinity when n tends to infinity. For each $n \geqslant 1$, we apply Itô's formula to a $\mathcal{C}^2$ function f_n that coincides with $x^{-\lambda}$ on $[\frac{1}{n}, +\infty[$:

$$\begin{aligned}
\mathcal{E}^{-\lambda}(M^n)_t &= 1 - \lambda \int_0^t \mathcal{E}^{-\lambda-1}(M^n)_{s^-} d\mathcal{E}(M^n)_s \\
&+ \frac{\lambda(\lambda+1)}{2} \int_0^t \mathcal{E}^{-\lambda-2}(M^n)_{s^-} d\left\langle (\mathcal{E}(M^n))^c, (\mathcal{E}(M^n))^c \right\rangle_s \\
&+ \sum_{s \leqslant t} \left[\mathcal{E}^{-\lambda}(M^n)_s - \mathcal{E}^{-\lambda}(M^n)_{s^-} + \lambda \mathcal{E}^{-\lambda-1}(M^n)_{s^-} \Delta \mathcal{E}^{-\lambda}(M^n)_s\right] \\
&= 1 + \int_0^t \mathcal{E}(M^n)_{s^-} dX_s^n,
\end{aligned}$$

where

$$X_t^n := -\lambda M_t^n + \frac{\lambda(\lambda+1)}{2} \langle (M^n)^c, (M^n)^c \rangle_t + \sum_{s \leqslant t} \left[(1 + \Delta M_s)^{-\lambda} - 1 + \lambda \Delta M_s\right],$$

and then $\mathcal{E}^{-\lambda}(M^n) = \mathcal{E}(X^n)$.

Let us define the non-truncated counterpart X of X^n:

$$X = -\lambda M + \frac{\lambda(\lambda+1)}{2} \langle M^c, M^c \rangle + W^{-\lambda}.$$

On the interval $[0, T_n[$, we have $X^n = X$ and $\mathcal{E}^{-\lambda}(M) = \mathcal{E}(X)$, now letting n tends to infinity, we obtain that $\mathcal{E}^{-\lambda}(M)$ and $\mathcal{E}(X)$ coincide on $[0, +\infty[$, which is the point (i) of the Lemma.

We want to use the proposition 5.6.1 to prove the point (ii), so we need to show that $\Delta A > -1$. We set

$$S = \inf\{t \geqslant 0 \ : \ \Delta A_t^{-\lambda} \leqslant \ -1\}.$$

It is a predictable time. Using this, and the fact that M and $(W^{-\lambda} - V^{-\lambda})$ are local martingales, we have

$$\Delta A_S^{-\lambda} = \mathbb{E}\left[\Delta A_S^{-\lambda} | \mathcal{F}_{S^-}\right] = \mathbb{E}\left[\Delta X_S | \mathcal{F}_{S^-}\right] = \mathbb{E}\left[(1 + \Delta M_S)^{-\lambda} | \mathcal{F}_{S^-}\right],$$

and since $\{S < +\infty\} \in \mathcal{F}_{S^-}$,

$$0 \geqslant I\!\!E\left[\mathbf{1}_{\{S<+\infty\}}(1 + \Delta A_S^{-\lambda})\right] = I\!\!E\left[\mathbf{1}_{\{S<+\infty\}}(1 + \Delta M_S)^{-\lambda}\right].$$

Then $\Delta M_S \leqslant -1$ on $\{S < +\infty\}$, which means that $S = +\infty$ and $\Delta A > -1$ a.s. The proof is now complete. ◻

We are finally in a position to state the Lemma on L^r integrability of exponential martingales for a negative exponent r.

Lemma 5.6.6. *Let $\lambda > 0$ and let M be a local martingale with bounded jumps, such that $\Delta M \geqslant -1 + \delta$, for a fixed $\delta > 0$, and $\langle M, M\rangle_t$ is bounded a.s. Then*

$$I\!\!E\left[\mathcal{E}(M)_t^{-\lambda}\right] < +\infty.$$

Proof. Let $n \geqslant 1$ be an integer. We will denote $\tilde{\mu}_M = \mu_M - \nu_M$ the compensated jump measure of M. Thanks to lemma 5.6.5, we write the decomposition

$$\mathcal{E}(M)^{-n} = \mathcal{E}(\tilde{N}^{-n})\mathcal{E}\left(\frac{1}{2}n(n+1) < M^c, M^c > +V^{-n}\right),$$

where $\tilde{N}^{-n}$ is a local martingale and V^{-n} is defined as $V^{-\lambda}$. Using Lemma 5.6.4, we have the inequality

$$V_t^{-n} \leqslant \int_0^t \int_E Cx^2 \nu_M(ds, dx)$$

and using the previous representation we obtain

$$\begin{aligned}
\mathcal{E}(M)_t^{-n} &\leqslant \mathcal{E}(\tilde{N}^{-n})_t \mathcal{E}\left(\frac{1}{2}n(n+1)\langle M^c, M^c\rangle + \int_0^\cdot \int_E Cx^2\nu_M(ds, dx)\right)_t \\
&\leqslant \mathcal{E}(\tilde{N}^{-n})_t \exp\left(\frac{1}{2}n(n+1)\langle M^c, M^c\rangle_t + \int_0^t \int_E Cx^2\nu_M(ds, dx)\right) \\
&\leqslant \mathcal{E}(\tilde{N}^{-n})_t \exp\left((\frac{1}{2}n(n+1) + C)\langle M, M\rangle_t\right) \\
&\leqslant C\mathcal{E}(\tilde{N}^{-n})_t \text{ since } \langle M, M\rangle_t \text{ is bounded.}
\end{aligned}$$

Let us prove now that the jumps of $\tilde{N}^{-n}$ are strictly bigger than -1. We compute

$$\begin{aligned}
\Delta\tilde{N}^{-n} &= \frac{\Delta N^{-n}}{1 + \Delta A^{-n}} \text{ where } A^{-n} \text{ is defined as in lemma 5.6.6} \\
&= \frac{(1 + \Delta M)^{-n}}{1 + \Delta V^{-n}} - 1 > -1 \text{ since } -1 < \Delta M \leqslant B \text{ and } \Delta V^{-n} > -1.
\end{aligned}$$

This implies that $\mathcal{E}(\tilde{N}^{-n})$ is a positive supermartingale which equals 1 at $t = 0$. We deduce

$$I\!\!E\left[\mathcal{E}(M)_t^{-n}\right] \leqslant CI\!\!E\left[\mathcal{E}(\tilde{N}^{-n})_t\right] \leqslant C.$$

We have the desired integrability for negative integers. We extend the property to any negative real number by Hölder's inequality. ◻

Chapitre 6

Numerical Implementation

6.1 Introduction

Avellaneda et al. [2] derived a pricing PDE (Avellaneda PDE aftermath) for uncertain volatility models. In practice, Avellaneda PDE is not solvable and one must rely on a finite difference scheme. But standard finite difference schemes can only be implemented when the number of variables - underlying assets or auxiliary variables - is small. For high dimensional case one needs to use Monte Carlo approach.

The Monte Carlo method is developed from a new advancement on the connection between fully nonlinear PDEs and second order backward stochastic differential equations (2BSDEs for short) presented in Cheridito et al. [22]. There exist three Monte Carlo schemes for UVM. Introduced with the first notion of 2BSDEs, Scheme Cheridito et al. [22] generalized the numerical method for solving classical BSDEs. Inspired by Scheme Cheridito et al., Fahim et al. [41] gave a new scheme without appealing to the theory of 2BSDEs. They proved the convergence of the scheme with a EDP approach. With UVM, the Avellaneda PDE for pricing is fully nonlinear. In this particular case, Scheme Guyon and Henry-Labordère [47] improved the two precedent ones without using the theory of 2BSDEs. For path-dependent options, these schemes can also be applied with some modifications and by using results obtained in Gobet et al. [45].

The main objective of this chapter is to study and implement the Scheme Guyon and Henry-Labordère.

6.2 Avellaneda pricing PDE

UVM were introduced by Avellaneda et al. [2], where the volatility process is only supposed to lie within an interval (it does not have a specific dynamic). And the value V of a derivative delivering some payoff $H_T(S_t, 0 \leqslant t \leqslant T)$ at maturity T is

$$V_t = \sup_{[t,T]} \mathbb{E}[H_T|\mathcal{F}_t]$$

where $\sup_{[t,T]}$ means that the supremum is taken over all $(\mathcal{F}_s)$-adapted processes $(\xi_s)_{t \leqslant s \leqslant T} \equiv \left((\sigma_s^\alpha, \rho_s^{\alpha\beta})_{1 \leqslant \alpha < \beta \leqslant d} \right)_{t \leqslant s \leqslant T}$ such that for all $s \in [t, T]$, ξ_s belongs to some compact domain D. Notice that the covariance matrix $\left(\rho^{\alpha\beta}\sigma^\alpha\sigma^\beta\right)_{1 \leqslant \alpha,\beta \leqslant d}$should be non-negative. We consider domains D of the form $D = [\underline{\sigma}, \bar{\sigma}]$ when $d = 1$, and $D = [\underline{\sigma}^1, \bar{\sigma}^1] \times [\underline{\sigma}^2, \bar{\sigma}^2] \times [\underline{\rho}, \bar{\rho}]$ when $d = 2$.

Applying stochastic control theory, the ask price can be presented by the solution of a fully nonlinear PDE.

In one-dimensional case, a risky asset follows a controlled diffusion under a risk-neutral measure

$$dS_t = \sigma_t S_t dW_t.$$

The valuation of an option can be written as the solution (in the viscosity sense) of an HJB equation with a control on the diffusion coefficient. This leads to a fully nonlinear second order PDE, the Avellaneda PDE.

$$\partial_t u(t,x) + \frac{1}{2}x^2 G(\partial_{xx}u(t,x))\partial_{xx}u(t,x) = 0,\ (t,x) \in [0,T) \times \mathbb{R}_+^*$$

with some terminal condition $u(T,x) = g(x)$, and $G(\Gamma) = \underline{\sigma}^2 1_{\Gamma<0} + \bar{\sigma}^2 1_{\Gamma \geqslant 0}$.

In d-dimensional case,

$$dS_t^\alpha = \sigma_t^\alpha S_t^\alpha dW_t^\alpha,\ dW_t^\alpha dW_t^\beta = \rho_t^{\beta\alpha} dt,\ 1 \leqslant \alpha < \beta \leqslant d.$$

where $\rho^{\alpha\beta}\sigma^\alpha\sigma^\beta_{1 \leqslant \alpha<\beta \leqslant d}$ is the non-negative covariance matrix.

For vanilla payoffs $H_T = g(S_T)$, where the payoff function g is assumed continuous with quadratic growth, we have the price $V_t = u(t, S_t)$ where $u(\cdot,\cdot)$ is the unique (viscosity) solution with quadratic growth of the following PDE

$$\partial_t u(t,x) + f(x, \nabla_{xx}u(t,x)) = 0,\ (t,x) \in [0,T) \times (\mathbb{R}_+^*)^d$$

with the terminal condition $u(T,x) = g(x)$ and the Hamiltonian

$$f(x,\Gamma) = \frac{1}{2} max_{(\rho^{\alpha\beta},\sigma^\alpha)_{1 \leqslant \alpha<\beta \leqslant d} \in D} \sum_{\alpha,\beta=1}^{d} \rho^{\alpha\beta}\sigma^\alpha\sigma^\beta x^\alpha x^\beta \Gamma^{\alpha\beta}.$$

6.3 Different schemes

In their paper introducing the 2BSDEs theory, Cheridito et al. provided a numerical scheme using Monte Carlo method to solve fully nonlinear PDEs. Fahim et al. then proved the convergence of a similar Monte Carlo scheme without appealing to the 2BSDEs. As presented in previous sections, Avellaneda PDE is a fully nonlinear PDE, so one can solve it numerically using both schemes. In the following, we derive this two schemes for this particular PDE.

1.With 2BSDEs theory (Cheridito et al.):
The 2BSDE associated to the Avellaneda PDE is

$$\begin{cases} dX_t^\alpha = \hat{\sigma}^\alpha X_t^\alpha dW_t^\alpha,\ \ dW_t^\alpha dW_t^\beta = \hat{\rho}^{\beta\alpha} dt,\ \ 1 \leqslant \alpha < \beta \leqslant d. \\ dY_t = -f(X_t,\Gamma_t)dt + \displaystyle\sum_{\alpha=1}^{d} Z_t^\alpha \circ \hat{\sigma}^\alpha X_t^\alpha dW_t^\alpha \\ dZ_t^\alpha = A_t^\alpha dt + \displaystyle\sum_{\beta=1}^{d} \Gamma_t^{\alpha\beta}\hat{\sigma}^\beta X_t^\beta dW_t^\beta \\ Y_T = g(X_T) \end{cases} \tag{6.3.1}$$

with some constant volatility $\hat{\sigma}^\alpha$ and some constant correlation $\hat{\rho}^{\alpha\beta}$ and where $\circ$ is the Stratonovich integral. Cheridito et al. proved that $Y_t = u(t, X_t)$, $Z_t = \nabla_x u(t, X_t)$, $\Gamma_t = \nabla_{xx} u(t, X_t)$ and $A_t = (\partial_t + \mathcal{L}^X)\nabla_x u(t, X_t)$. Then we have $Y_0 = u(0, X_0) = u(0, S_0) = V_0$.

By discretizing the continuous processes of the 2BSDE and taking the conditional expectation of both sides of equations (resp. first multiplying both sides by Brownian increment ΔW, then taking the conditional expectation), we can compute the price Y (resp. the delta Z and the gamma Γ) backwardly. The following is the complete scheme deriving from 2BSDEs theory.

Scheme Cheridito et al.:

$$\begin{cases} Y^\Delta_{t_n} = g(X^\Delta_{t_n}),\ Z^\Delta_{t_n} = \nabla g(X^\Delta_{t_n}) \\ Y^\Delta_{t_{i-1}} = \mathbb{E}_{i-1}\left[Y^\Delta_{t_i}\right] \\ \quad + \left(f\left(t_{i-1}, X^\Delta_{t_{i-1}}, Y^\Delta_{t_{i-1}}, Z^\Delta_{t_{i-1}}, \Gamma^\Delta_{t_{i-1}}\right) - \frac{1}{2} tr\left[\left(\hat{\sigma} X^\Delta_{t_{i-1}}\right)\left(\hat{\sigma} X^\Delta_{t_{i-1}}\right)' \Gamma^\Delta_{t_{i-1}}\right]\right) \Delta t_i \\ Z^\Delta_{t_{i-1}} = \frac{1}{\Delta t_i}\left(\hat{\sigma} X^\Delta_{t_{i-1}}\right)^{\prime -1} \mathbb{E}_{i-1}\left[Y^\Delta_{t_i}\Delta W_{t_i}\right] \\ \Gamma^\Delta_{t_{i-1}} = \frac{1}{\Delta t_i}\mathbb{E}_{i-1}\left[Z^\Delta_{t_i}\Delta W'_{t_i}\right]\left(\hat{\sigma} X^\Delta_{t_{i-1}}\right)^{-1} \end{cases} \tag{6.3.2}$$

2.Without 2BSDEs theory (Fahim et al.):

We can rewrite $\partial_t u(t, x) + f(x, \nabla_{xx} u(t, x)) = 0,\ (t, x) \in [0, T) \times (\mathbb{R}^*_+)^d$ as

$$\begin{aligned} &\partial_t u(t,x) + \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \hat{\rho}^{\alpha\beta}\hat{\sigma}^\alpha\hat{\sigma}^\beta x^\alpha x^\beta \left(\nabla_{xx}u(t,x)\right)^{\alpha\beta} \\ &\qquad + \left(f(x, \nabla_{xx}u(t,x)) - \sum_{\alpha,\beta=1}^{d} \hat{\rho}^{\alpha\beta}\hat{\sigma}^\alpha\hat{\sigma}^\beta x^\alpha x^\beta \left(\nabla_{xx}u(t,x)\right)^{\alpha\beta} \right) = 0 \end{aligned}$$

Let us denote

$$F(x, \nabla_{xx}u(t,x)) = f(x, \nabla_{xx}u(t,x)) - \sum_{\alpha,\beta=1}^{d} \hat{\rho}^{\alpha\beta}\hat{\sigma}^\alpha\hat{\sigma}^\beta x^\alpha x^\beta \left(\nabla_{xx}u(t,x)\right)^{\alpha\beta}$$

and

$$dX^\alpha_t = \hat{\sigma}^\alpha_t X^\alpha_t dW^\alpha_t,\ dW^\alpha_t dW^\beta_t = \hat{\rho}^{\beta\alpha}_t dt,\ 1 \leqslant \alpha < \beta \leqslant d.$$

a log-normal dynamics with constant $\hat{\sigma}^\alpha$ and constant $\hat{\rho}^{\alpha\beta}$. Assuming that the Avellaneda PDE has a classical solution, it follows from Itô's formula that

$$\mathbb{E}_{t_i,x}\left[u\left(t_{i+1}, X_{t_{i+1}}\right)\right] = u\left(t_i, x\right) + \mathbb{E}_{t_i,x}\left[\int_{t_i}^{t_{i+1}} (\partial_t + \mathcal{L}^X) u\left(t, X_t\right) dt\right]$$

Since u solves the above PDE, this provides

$$\mathbb{E}_{t_i,x}\left[u\left(t_{i+1}, X_{t_{i+1}}\right)\right] = u\left(t_i, x\right) - \mathbb{E}_{t_i,x}\left[\int_{t_i}^{t_{i+1}} F(X_t, \nabla_{xx} u\left(t, X_t\right)) dt\right]$$

By approximating the integral $\int_{t_i}^{t_{i+1}} F(X_t, \nabla_{xx} u\left(t, X_t\right)) dt$ and applying the Malliavin calculus for the second derivative Γ, one can also derive a similar scheme without appealing to the 2BSDEs theory. The following is the complete scheme.

Scheme Fahim et al.:

$$\begin{cases} Y^{\Delta}_{t_n} = g(X^{\Delta}_{t_n}) \\ Y^{\Delta}_{t_{i-1}} = \mathbb{E}_{i-1}\left[Y^{\Delta}_{t_i}\right] \\ \quad + \left(f\left(t_{i-1}, X^{\Delta}_{t_{i-1}}, Y^{\Delta}_{t_{i-1}}, Z^{\Delta}_{t_{i-1}}, \Gamma^{\Delta}_{t_{i-1}}\right) - \frac{1}{2} tr\left[\left(\hat{\sigma} X^{\Delta}_{t_{i-1}}\right)\left(\hat{\sigma} X^{\Delta}_{t_{i-1}}\right)' \Gamma^{\Delta}_{t_{i-1}}\right]\right) \Delta t_i \\ \Gamma^{\Delta}_{t_{i-1}} = \mathbb{E}_{i-1}\left[Y^{\Delta}_{t_i}\left(\left(\hat{\sigma} X^{\Delta}_{t_{i-1}}\right)'\right)^{-1} \frac{\Delta W_{t_i}\left(\Delta W'_{t_i} - \Delta t_i I_d\right)}{\Delta t_i^2}\left(\hat{\sigma} X^{\Delta}_{t_{i-1}}\right)^{-1}\right] \end{cases} \quad (6.3.3)$$

In the particular case of UVM, based on the Scheme Fahim et al., by taking arbitrary constant volatility $\hat{\sigma}$ and correlation $\hat{\rho}$ to simulate the process (thus a Black-Scholes model) and applying the Malliavin calculus for a log-normal diffusion, one can have a new scheme for Avellaneda PDE.

Scheme Guyon and Henry-Labordère:

$$\begin{cases} X^{\alpha}_{t_i} = X^{\alpha}_0 e^{-(\hat{\sigma}^{\alpha})^2 \frac{t_i}{2} + \hat{\sigma}^{\alpha} W^{\alpha}_{t_i}}, \ \mathbb{E}\left[\Delta W^{\alpha}_{t_i} \Delta W^{\beta}_{t_i}\right] = \hat{\rho}^{\alpha\beta} \Delta t_i \\ Y_{t_n} = g(X_{t_n}) \\ Y_{t_{i-1}} = \mathbb{E}\left[Y_{t_i} | X_{t_{i-1}}\right] + \left(f\left(X_{t_{i-1}}, \Gamma_{t_{i-1}}\right) - \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \hat{\rho}^{\alpha\beta} \hat{\sigma}^{\alpha} \hat{\sigma}^{\beta} X^{\alpha}_{t_{i-1}} X^{\beta}_{t_{i-1}} \Gamma^{\alpha\beta}_{t_{i-1}}\right) \Delta t_i \\ \left(\Delta t_i\right)^2 \hat{\sigma}^{\alpha} \hat{\sigma}^{\beta} X^{\alpha}_{t_{i-1}} X^{\beta}_{t_{i-1}} \Gamma^{\alpha\beta}_{t_{i-1}} = \mathbb{E}\left[Y_{t_i}\left(U^{\alpha}_{t_i} U^{\beta}_{t_i} - \Delta t_i \hat{\rho}^{-1}_{\alpha\beta} - \Delta t_i \hat{\sigma}^{\alpha} U^{\alpha}_{t_i} \delta_{\alpha\beta}\right) | X_{t_{i-1}}\right] \end{cases} \quad (6.3.4)$$

with $U^{\alpha}_{t_i} \equiv \sum_{\beta=1}^{\alpha} \hat{\rho}^{-1}_{\alpha\beta} \Delta W^{\beta}_{t_i}$.

Notice that for the Avellaneda PDE the coefficient f depends only on the second derivative of the solution (Γ), so there is no need to compute the first derivative (Z), then Scheme Fahim et al. and Scheme Guyon and Henry-Labordère should be more efficient than Scheme Cheridito et al. with which one always needs the first derivative in order to obtain the second derivative.

Furthermore, there are also other differences between the three schemes:

In Scheme Cheridito et al. they discretized the continuous process of the Gamma Γ. In Scheme Fahim et al. they used the Gamma Malliavin weight for the Bachelier model. In Scheme Guyon and Henry-Labordère they use explicitly the Malliavin weight for a log-normal diffusion with constant volatility $\hat{\sigma}$ and correlation $\hat{\rho}$.

With Scheme Fahim et al., the forward diffusion process is simulated by Euler scheme while with Scheme Guyon and Henry-Labordère the diffusion is simulated exactly. And for this reason, there is difference in computing Gamma Γ for these two schemes.

In the particular cases of no volatility uncertainty or of convex or concave European payoffs, the nonlinear PDE reduces to a (classical) Black-Scholes pricing PDE and Scheme Guyon and Henry-Labordère is exact, contrary to Scheme Cheridito et al. and Scheme Fahim et al.. Also Scheme Guyon and Henry-Labordère can be applied for discontinuous payoffs.

6.4 Approximation of Conditional Expectations

The most important part in all three schemes is the approximation of conditional expectations. With Scheme Guyon and Henry-Labordère, there are $\frac{d(d+1)}{2}+1$ expectations to compute at each discrete backward date, one for price Y and the other $\frac{d(d+1)}{2}$ for Gamma Γ where d is the number of underlying assets.

There exits several ways to approximate conditional expectations as for the pricing of Bermuda options.

1. One can use parametric regression as in the Longstaff-Schwartz methods (see Gobet et al. [45] for details in the case of BSDEs):

$$\mathbb{E}\left[Y_{i+1}|X_i=x\right]\approx\sum_{k=1}^{N}c_kp_k(x)$$

2. For low dimensional case, one can also use non-parametric regression:

$$\mathbb{E}[Y_{i+1}|X_i=x]\approx\frac{\mathbb{E}[Y_{i+1}\delta_N(X_i-x)]}{\mathbb{E}[\delta_N(X_i-x)]}$$

   with $\delta_N(\cdot)$ a kernel approximating a Dirac mass at zero.

3. Another possibility is to use Malliavin's weight (see Bouchard and Touzi [16] in the case of BSDEs and Bouchard and Warin [18] for Bermuda options).

Since parametric regression is the most appropriate for high dimensional case, we choose the technique which is similar to the Longstaff-Schwartz Monte Carlo regression. For possible improvements, one can try all techniques then compare results and choose perhaps the best technique.

When there are several backward dates, who represent the discretization points of the time dimension for a continuous process, the regression error cumulates from a date to another, thus a non accurate approximation can deteriorate the pricing quality. From our different tests we can see that the conditional expectation for Y can be approximated well by regression. When σ_{min} is equal to σ_{max}, the Avellaneda model is reduced to the simple Black-Scholes model then the option price does not depend on Gamma Γ. In this case, we have a good precision for the option price. That means a good approximation of the conditional expectation for Y. When σ_{min} is different from σ_{max}, the option price does depend on Gamma Γ and becomes less precise, this could mean that the approximation of the conditional expectations for Gamma is not as good as for Y. This can be explained by the fact that the Gamma Γ simulated with Malliavin calculus has large variance. There are many active studies in variance reduction techniques for computing Greeks. Several methods exist such as localization and importance sampling. We don't apply these techniques here. In the real life pricing, these techniques could be used to have a better precision for Gamma thus for price.

In conclusion, the pricing precision depends essentially on quality of approximation of conditional expectations by regression , particularly for Gamma Γ.

As presented in Bouchard and Warin [18], there are two main questions for the approximation by regression. First is the choice of the regression procedure which refers to numerical algorithm to solve the system $A\alpha = B$ for α and second the choice of regression basis functions.

Regression procedure

We can use the following techniques.

1. Choleski decomposition LL' of $A'A$: In this case, one solves $LL'\alpha = A'B$. It is not memory consuming because the A matrix does not need to be constructed. This algorithm is the most efficient but not stable.

2. QR decomposition of A as QR: One solves $R\alpha = Q'B$. This technique is more stable but much more time consuming. The A matrix has to be stored.

3. Singular Value Decomposition (SVD) of A as UWV' : One has $\alpha = V\left[diag\left[1/w_i\right]\right]U'B$. It is the most stable among these three techniques. However, this method suffers the same problem as the QR algorithm in term of memory needed to create the matrix A and is the most time consuming.

Basis functions

1. Polynomial.

   This kind of function basis is very easy to implement in practice, but it has a major flaw. It is difficult to find an optimal degree of the functional basis. Besides, an increase in the number of basis functions often leads to a deterioration in the accuracy of the result. This is due to extreme events that the polynomials try to fit, leading to some bad representation of the function.

   From our numerical tests, we see that the choice of maximal degree for polynomial basis can affect the results. And in general, with numerical experimentations and good understanding of the financial product to price, we can find a suitable maximal degree with which this function basis works well.

   Note that, in the case where an explicit formula is available for the corresponding European option, one can replace the estimator $\hat{\mathbb{E}}[Y_{t_{i+1}}|\mathcal{F}_{t_i}]$ in algorithms by $\hat{\mathbb{E}}[Y_{t_{i+1}} - P^{euro}(t_{i+1}, X_{t_{i+1}})|\mathcal{F}_{t_i}] + P^{euro}(t_i, X_{t_i})$ where $P^{euro}(t,x)$ denotes the price of the corresponding European option at time t if $X_t = x$. This is similar to control variates technique for variance reduction. The idea behind this comes from the fact that the European price process (discounted) $P^{euro}(\cdot, X)$ is a martingale, and that it generally explains a large part of the price. Alternatively, $P^{euro}(t_i, \cdot)$ could also be included in the regression basis.

2. Calls with different strikes.

   In practice, one possibility is to regress on options of the underlying that are very similar to the payoffs we are trying to price. It seems obvious that the more an option looks like the option we are pricing, the more it will contains information about the price. The theoretical perfect case is for example when we are trying to price a Call and when that Call is in the regression basis, we just need one basis function. When payoffs are Calls combination, we can regress on a base of Calls with different strikes centered around the money-strike.

3. Hypercubes (Adaptative local basis approach)

   The idea is to use, at each time step t_i, a set of functions ψ_q (for instance, polynomial with maximal degree 1 or 2), $q \in [0, M_M]$ having local hypercube support $D_{i_1, i_2, \dots, i_d}$ where $i_j = 1$ to I_j, $M_M = \prod_{k=1,\cdots,d} I_k$. With this approximation we do not assure the continuity of the approximation. It has the advantage to be able to fit any , even discontinuous, function. In order to avoid oscillations, the support are chosen so that they contain roughly the same number of particles. When using such local functions, it is possible to use the Choleski method, which is the most efficient for solving the regression problem.

Given that the main objective for us is to have a stable pricing algorithm so we choose the more stable procedure SVD although it's time and memory consuming. Indeed, for the next step, we can effectively choose the regression procedure according to the choice of basis functions. This should make the pricing procedure more efficient.

And we notice that different basis functions can be used for approximating price Y and Gamma Γ, for instance polynomial for Y and local support basis (hypercube) for Γ. That could improve the option price precision in some cases. The idea behind this is that Y and Γ have different forms , one kind of basis functions may be a good choice for Y but a bad one for Γ. So well understanding of products is important for a efficient application of the scheme.

6.5 Forward Monte Carlo Pricing Step

From our numerical experiments, we see that the algorithm presented above produces an unpredictable bias (lower or higher).

As suggested in Guyon and Henry-Labordère [47], one can have a lower price by adding a forward simulation step. This is a commonly used technique for the pricing of American options with Monte Carlo method. In this step, the optimal volatility is determined by the function of Gamma computed in the first step.

In order to build a low-biased estimate, one can simulate another set of replicas of X

$$dX_t^\alpha = \sigma_t^{*\alpha} X_t^\alpha dW_t^\alpha \quad dW_t^\alpha dW_t^\beta = \rho_t^{*\beta\alpha} dt \quad 1 \leqslant \alpha < \beta \leqslant d$$

in an independent second Monte Carlo procedure, where the simulated optimal volatility and correlation, $\sigma_t^{*\alpha}$ and $\rho_t^{*\alpha\beta}$, are the solutions to

$$max_{(\rho^{\alpha\beta}, \sigma^\alpha)_{1 \leqslant \alpha < \beta \leqslant d} \in D} \sum_{\alpha,\beta=1}^{d} \rho^{\alpha\beta} \sigma^\alpha \sigma^\beta x^\alpha x^\beta \psi(t, X_t)$$

with ψ the approximation for Gamma Γ of the first backward step. Because the covariance matrix is suboptimal, the obtained estimator is low-biased. One can run the second Monte Carlo simulation with more paths and a (much smaller) time step for the forward discretization of X. Since the estimator is low-biased, the true price is larger than each of the simulated prices.

When the bias is unknown, one cannot make such a claim and it is hard to guess where the true price is. But the low-biased price could be imprecise like in the case of Call

Sharpe options and generally of path-dependent options. Also we know that Monte Carlo simulated price lies in a confidential interval including the true price for a fixed number of paths (even very large). Since the second step is a Monte Carlo pricing method, then the simulated price can be larger than the true price. So we have to be cautious with this low-biased price. In general, we prefer the low-biased price for European options and the backward price for path-dependent ones. In the real life pricing, we'd better compute both low-biased and backward prices. Then if theses prices are very different from each other, this could mean that the low-biased price does not have a good precision.

Furthermore, in the forward step, there is another problem. From the backward step, we get the estimation of Gamma Γ only for $N-1$ of the N time-intervals (there is no regression performed for the first time interval). It means that we have to take an arbitrary fixed value of volatility $\hat{\sigma}$ (for example the mid-volatility) for the first time interval. If this period is large which is the case when there are 2 or 4 backward dates, then the error induced by this arbitrary choice could be important. To fix this problem, we propose to add an extra backward date close to the initial date in the first step. By doing so, we may introduce more regression approximation error, but we reduce the size of time interval where an arbitrary volatility is used, so the error induced by non-optimal volatility. The better results show that with a large number of simulations, the added regression error is small compare to the reduced non-optimal volatility error.

6.6 Pricing with path-dependent variables

When the price of an option depends on path-dependent variables A (can be average, max, min, realized variance) whose values can change only at discrete dates (fixing dates), one solves Avellaneda PDE between two such discrete dates t_l and t_{l-1} for fixed values of the path-dependent variables A, and defines

$$u(t^-_{t_{l-1}}, X, A) = u(t^+_{t_{l-1}}, X, \phi(A))$$

with the function ϕ linking the past and new values of the path-dependent variables on each fixing date.

For instance, if the option value depends on a monthly-computed realized variance, then

$$A^1_t = \sum_{\{l|t_l \leqslant t\}} \left(ln \frac{X_{t_l}}{X_{t_{l-1}}} \right)^2, \; A^2_t = X_{sup_{\{l|t_l \leqslant t\}} t_l}, \phi(A) = A^1 + \left(ln \frac{X}{A^2} \right)^2.$$

In our numerical experiments, we apply the Monte Carlo scheme to price Asian options and Call Sharpe options with UVM.

In real-life contracts, Asian options are in fact defined in terms of *discretely sampled average*, like

$$A = \frac{T}{n} \sum_{i=1}^{n} X_{t_i}.$$

Let us introduce the process Y such that

$$Y(t_i \leqslant t < t_{i+1}) = \frac{1}{i} \sum_{k=1}^{i} X_{t_k}.$$

The factor process (X, Y) is Markovian in the risk-neutral Black-Scholes model for X, with related generator $\mathcal{G}_{X,Y}v(X, Y)$ given by the usual Black-Scholes generator $\mathcal{G}_X v(X, Y)$ on each time interval (t_{i-1}, t_i). Moreover, one has

$$A = TY_T.$$

We proceed backwardly within each time interval (t_{i-1}, t_i) as in the case of European options with a terminal condition obtaining by the continuation condition at the fixing date t_i,

$$\begin{cases} \partial_t v_i + \dfrac{1}{2}\sigma^2 X^2 \partial^2_{X^2} v_i = 0 \\ v_i(t_{i+1}, X, Y) = v_{i+1}(t_{i+1}, X, Y_+) \end{cases} \tag{6.6.1}$$

where Y_+ is obtained via the following jump conditions at the monitoring date t_{i+1}:

$$Y_+ = \frac{i}{i+1}Y + \frac{X}{i+1}.$$

Indeed, the cost of solving the above PDE is essentially that of solving M one-dimensional PDE problems, where M is a generic number of mesh points for average dimension.

We implement two algorithms for path-dependent options derived from Scheme Guyon and Henry-Labordère for non-path-dependent ones presented in previous sections. The first one is inspired by the finite difference method. That mean we subdivide the path-dependent variable (discrete arithmetic average for Asian options and discrete realized variance for Call Sharpe). We know that between two discrete fixing dates, the path-dependent variable does not change. So we use the scheme within such period taking price at latest fixing date as terminal condition. And at each discrete fixing date, we compute price depending on the path-dependent variable of the previous date using the continuation condition.

But this algorithm is very time and memory consuming. Because Scheme Guyon and Henry-Labordère is applied to each subdivided value of path-dependent variable. And if there are several discrete fixing dates, we need to simulate a large number of paths to have a good convergence for each subdivided value. On average, 50000 simulated paths are needed and if there are 100 subdivided values, then there will be $100 \times 50000 = 5000000$ simulations. That takes a lot of time and memory. Instead the second algorithm is a purely Monte Carlo method which is inspired by Gobet et al [45]. With this method, in order to approximate conditional expectations, we use spot price and path-dependent variable value to construct regression basis functions. Thus we need to simulate the path-dependent variable value at each fixing date. It is also possible to construct regression basis functions with only spot price. This approximation induces some additional error, but it's easier to implement and takes less time to execute. If we use path-dependent variable to construct basis functions, then for different payoffs, we need different form of basis function. For instance, the polynomial basis with spot price and path-dependent variable works for pricing Asian options but is not well adapted for Call Sharpe options, since the payoff for a Sharpe option is the ratio of an European payoff and the realized standard deviation (which is the path-dependent variable in this case). So polynomial

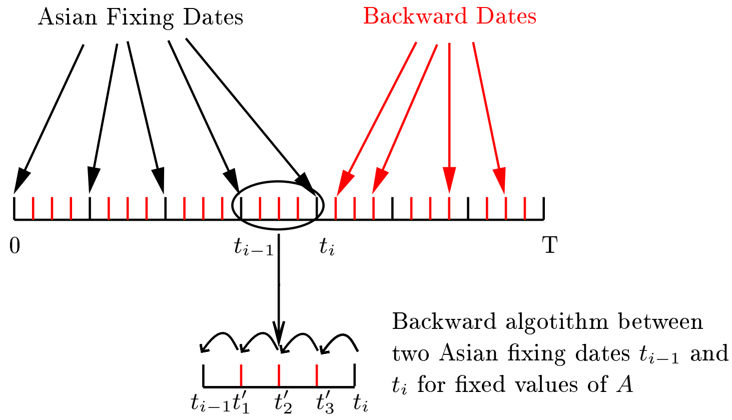

Figure 6.1: 2BSDE scheme for path-dependent options pricing

function basis constructed with the ratio of spot price and realized standard deviation should work better. However one need to pay attention while using the ratio, since the realized standard deviation may be very small for fixing dates close to the initial pricing date. Then the ratio may become very big which deteriorates the quality of approximation of conditional expectations by regression and give irrational option price. We do some tests with both algorithms and with different basis functions.

For path-dependent options as Asian ones or Call Sharpe, there is also a problem in the forward Monte Carlo simulation step. For the Finite Difference-like Monte Carlo scheme, the regression function for Gamma Γ will depend on the path-dependent variable, so we need to stock the regression parameters in a 5-dimensional variable. And in the forward step, we need to simulate the path-dependent variable to estimate Gamma Γ. With the purely Monte Carlo method where we simulate the path-dependent variables, the forward step does not perform very well neither. Imagine that within one time interval, the paths are simulated with a sub-optimal volatility, then this sub-optimality will persist until the maturity, so lead to bad estimation of the path-dependent variable thus to severe mis-pricing. Therefore, we prefer the backward price for path-dependent options.

When the path-dependent variables values change continuously, it is possible treat these variables like processes by adding some diffusion terms.

For an option depending on continuous path-dependent variables , the Hamiltonian f may not involve only the Gammas. For instance, in the single-asset case, if the price $u(t,x,v)$ of an option depends on the continuously compounded realized variance v, the Hamiltonian reads

$$f(x,\partial_x^2 u,\partial_v u) = max_{\underline{\sigma}\leqslant\sigma\leqslant\bar{\sigma}}\sigma^2\left(\frac{1}{2}x^2\partial_x^2 u+\partial_v u\right),$$

i.e., the optimal volatility is either $\underline{\sigma}$ or $\bar{\sigma}$, depending on the sign, not of the Gamma $\partial_x^2 u$, but of $\frac{1}{2}x^2\partial_x^2 u+\partial_v u$.

The 2BSDEs approach can easily be adapted to the case when the realized variance V_t changes continuously. One can show that in this case the price of the option with UVM can be written $u(t,X_t,V_t)$ where u is solution to

$$\partial_t u + f(x,\partial_x^2 u(t,x,v),\partial_v u(t,x,v)) = 0$$

Then one can associate a two-dimensional 2BSDE on the (X, V) plane to this fully nonlinear PDE:

$$\begin{cases} dX_t = \hat{\sigma} X_t dW_t^0 \\ dV_t = \hat{\sigma}^2 dt + \eta dW_t^1 \\ dY_t = (-f(X_t, \Gamma_t^{XX}, Z_t^V) + \mathcal{L}^{X,V} u(t, X_t, V_t))dt + Z_t^X \hat{\sigma} X_t dW_t^0 + Z_t^V \eta dW_t^1 \end{cases} \tag{6.6.2}$$

with

$$\mathcal{L}^{X,V} = \frac{1}{2}\hat{\sigma}^2 (X^2 \Gamma^{XX} + 2Z^V) + \frac{1}{2}\eta^2 \Gamma^{VV}.$$

In Guyon and Henry-Labordère [47], they used $\hat{\sigma}^2$ as the (forward) drift for the variance V, but this is arbitrary. They have introduced a diffusion term for V_t. Here η is a constant and W^1 a Brownian motion orthogonal to W^0. Adding this purely numerical volatility term allows to compute $Z_t^V = \partial_V u$. Just as the solution u of the PDE, the 2BSDE is independent of η, but the numerical scheme depends on it. A too small or too large value for η would lead to a bad regression-based estimation of Z_t^V.

For Asian options with continuous average, the factors are X (price) and $I = \int_0^{\cdot} X_t dt$, the running time-average of X. With the Black-Scholes model, the pair (X, I) is then a Markov process with generator $\mathcal{G}_{X,I}$ given by

$$\mathcal{G}_{X,I} = \frac{1}{2}\sigma^2 X^2 \partial_{X^2}^2 + X\partial_I.$$

Note that the generator $\mathcal{G}_{X,I}$ is degenerate in the I variable.

Then the pricing problem writes:

$$\begin{cases} \partial_t v + \mathcal{G}_{X,I} v = 0 \\ v(T, X, I) = \phi(X, I) \end{cases} \tag{6.6.3}$$

Note that the numerical resolution of the above PDE requires special care to cope with the degeneracy of the generator in the I variable (PDE in dimension $1\frac{1}{2}$).

Alternatively to the previous approach, it is possible to reduce the pricing problem to a one-dimensional PDE easier to solve numerically, by working in the numeraire X. The price is $X_t u(t, \eta_t)$, $t \in (0, T]$ where $\eta_t = \frac{1}{X_t}\left(K - \frac{I_t}{T}\right)$ and u is the solution of the following one-dimensional PDE:

$$\begin{cases} \partial_t u + \frac{1}{T}\partial_\eta u + \frac{1}{2}\sigma^2 \eta^2 \partial_{\eta^2}^2 u = 0 \\ u(T, \eta) = \eta^+ \end{cases} \tag{6.6.4}$$

For Asian options, we can take a similar approach as for Call Sharpe:

$$\begin{cases} \partial_t u + x\partial_A u + max_{\underline{\sigma} \leqslant \sigma \leqslant \bar{\sigma}} \sigma^2 \left(\frac{1}{2} x^2 \partial_x^2 u\right) = 0 \\ u(T, x, A) = \phi(A) \end{cases} \tag{6.6.5}$$

We can associate a two-dimensional 2BSDE on the (X, A) plane to this fully nonlinear PDE:

$$\begin{cases} dX_t = \hat{\sigma} X_t dW_t^0 \\ dA_t = X_t dt + \eta dW_t^1 \\ dY_t = (-f(X_t, \Gamma_t^{XX}) + \mathcal{L}^{X,A} u(t, X_t, A_t))dt + Z_t^X \hat{\sigma} X_t dW_t^0 + Z_t^A \eta dW_t^1 \end{cases} \tag{6.6.6}$$

with

$$\mathcal{L}^{X,A}u(t,X_t,A_t) = \frac{1}{2}\eta^2\Gamma_t^{AA} + \frac{1}{2}\hat{\sigma}^2 X_t^2\Gamma^{XX}$$

As presented in Gobet et al. [45], one can probably also use an discrete approximation for the continuous running-time average, then apply the same approach as for the case of discrete average. This is a better alternative than having to solve the two-dimensional degenerate PDEs (problems in dimension $1\frac{1}{2}$, unless specific dimension reduction techniques are available).

6.7 Numerical Experiments

The final meta-algorithm for pricing can be summarized in the following steps:

1. Simulate N_1 replicas of X with a log-normal diffusion on discrete dates t_k, $k = 1, \dots, M_1$.

   More precisely, from the initial spot price X_0, generate N_1 paths of X with a time step for discretization $\Delta t = T/M_1$.

   $$X_{t_{k+1}}^{M_1} = X_{t_k}^{M_1} exp\left\{-\frac{1}{2}\hat{\sigma}^2\Delta t_k + \hat{\sigma}\Delta W_{t_k}\right\}$$

   For the path-dependent options, path-dependent variables are simulated with these realizations of X.

2. Apply the backward algorithm Scheme Guyon and Henry-Labordère using a regression approximation.

   Calculate the maturity payoffs Y_T for all N_1 simulated paths. Then compute backwardly from $k = M_1 - 1$ to $k = 1$.

   At each t_k, construct the explanatory matrix A_k with basis functions $p_0, \cdots, p_{l-1}$

   $$A_k = \begin{pmatrix} p_0(X_{t_k}^{N_1,1}) & p_1(X_{t_k}^{N_1,1}) & \cdots & p_{l-1}(X_{t_k}^{N_1,1}) \\ p_0(X_{t_k}^{N_1,2}) & p_1(X_{t_k}^{N_1,2}) & \cdots & p_{l-1}(X_{t_k}^{N_1,2}) \\ \vdots & \vdots & \vdots & \vdots \\ p_0(X_{t_k}^{N_1,N_1}) & p_1(X_{t_k}^{N_1,N_1}) & \cdots & p_{l-1}(X_{t_k}^{N_1,N_1}) \end{pmatrix}$$

   A_k has N_1 rows and l columns, where l is the number of basis functions.

   Perform the regression for Γ and Y on the columns of A_k, then compute Γ_{t_k} and Y_{t_k} by formulas in the equation (6.3.4) where the conditional expectations are replaced by their approximations

   $$\mathbb{E}[\cdot|X] = \sum_{i=0}^{l-1} k_i p_i(X).$$

   From $t = t_1$ to $t = t_0$, there is no regression to perform since the conditional expectations $\mathbb{E}[\cdot|\mathcal{F}_0]$ are indeed expectations $\mathbb{E}[\cdot]$. Apply always the scheme to the price Y and Gamma Γ

$$(\Delta t_1)^2 \hat{\sigma}^\alpha \hat{\sigma}^\beta X_{t_0}^\alpha X_{t_0}^\beta \Gamma_{t_0}^{\alpha\beta} = \mathbb{E}\left[Y_{t_1}\left(U_{t_1}^\alpha U_{t_1}^\beta - \Delta t_1 \hat{\rho}_{\alpha\beta}^{-1} - \Delta t_1 \hat{\sigma}^\alpha U_{t_1}^\alpha \delta_{\alpha\beta}\right)\right]$$

$$Y_{t_0} = \mathbb{E}\left[Y_{t_1}\right] + \left(f\left(X_{t_0}, \Gamma_{t_0}\right) - \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \hat{\rho}^{\alpha\beta}\hat{\sigma}^\alpha\hat{\sigma}^\beta X_{t_0}^\alpha X_{t_0}^\beta \Gamma_{t_0}^{\alpha\beta}\right)\Delta t_1.$$

This is the end of the backward step.

3. Simulate N_2 independent replicas of X on discrete dates t_k, $k = 1 \dots M_2$ using the Gamma functions computed at the previous step then compute the mean of payoffs on maturity $t_{M_2} = T$. For the forward step, it's a classic Monte Carlo method with N_2 simulations and M_2 discrete dates.

It is noteworthy that the scheme presented previously have 3 key convergence parameters (N_2 and M_2 being fixed in our tests): the number of time steps for discretization M_1; the basis functions; the number of simulations N_1; Besides the backward step diffusion volatility $\hat{\sigma}$ can also influence the pricing precision.

As M_1 becomes large, which means the time step Δt becomes small, we need more and more simulations (increasing N_1) to obtain an accurate price. Gobet et al. [45], for BSDEs, and A. Fahim et al. [41] for fully nonlinear PDEs, also noticed that the numerical scheme diverges when the time step Δt goes to zero, the number of simulations N_1 being fixed.

A kind of Picard iterations method can also be applied to reduce the pricing error. Before proceeding to Step 3, we may repeat Steps 1 and 2, replacing $(\hat{\rho}^{\alpha\beta}\hat{\sigma}^\alpha\hat{\sigma}^\beta)$ by the optimal covariance matrix estimated at Step 2. This should improve the precision of lower bound for the price in Step 3.

In our numerical experiments, we take $T = 1$, and, for each asset α, $X_0^\alpha = 100$, $\underline{\sigma}^\alpha = 0.1$, $\bar{\sigma}^\alpha = 0.2$ and we use the constant mid-volatility $\hat{\sigma}^\alpha = 0.15$ (it will be mentioned if other values are used) to generate the first N_1 replicas of X. We also pick $t_i = i/n$, so that $\Delta = 1/n$. In the forward Monte Carlo pricing step (contrary to the backward step where Gamma Γ is calibrated), the $N_2 = 50000$ replicas of X use a time step $\Delta_2 = 1/52$.

European Call. First, let us test our algorithm in the case of an European Call option with payoff $(X_T - K)^+$. We take $K = 100$. The true BS price is $C_{BS} = 7.97$. As showed on the following figure, the algorithm with backward step produces an unpredictable bias.

By adding the forward step, the prices obtained are low-biased. For a Call option, these prices have good precision with a small number of simulations.

We know that theoretically the pricing of European Call option with UVM depends only on the maximal volatility $\bar{\sigma}$, because the payoffs is convex and the Gamma Γ is always positive. But as we show in the tests, the numerical result depends on the minimal volatility $\underline{\sigma}$. The reason is that the simulated Gamma Γ may take negative value in numerical experience, then during the forward Monte Carlo simulation step, in some time intervals paths are generated with the minimal volatility $\underline{\sigma}$, so simulated prices may be much lower than the true BS price when $\underline{\sigma}$ is very small. We can also see that increasing simultaneously the backward dates number and the simulated paths number can diminish

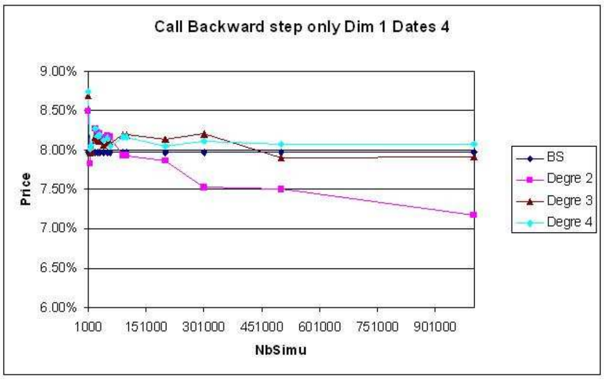

Figure 6.2: European Call Pricing with Backward step only Vol min=10% Vol max=20% for polynomial basis

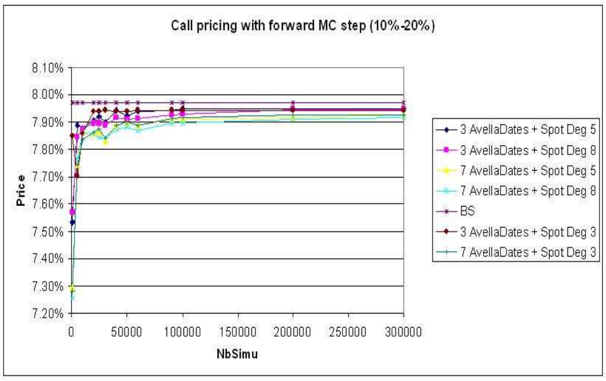

Figure 6.3: Call pricing with Forward step for different backward dates and polynomial basis functions

this difference. But this will increase considerably the computation time. So an advice for the use of this pricing algorithm is varying different parameters and taking the maximum of simulated prices (because these prices are low-biased).

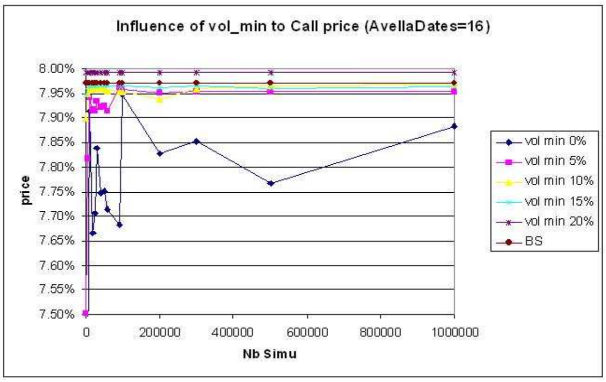

Figure 6.4: Influence of Vol Min to Call pricing Vol max=20%

For an European Call, the Gamma Γ is positive, so the optimal volatility σ^* is always equal to $\bar{\sigma}$. To test our algorithm, a more interesting case is European Call Spread whose Gamma Γ can be positive or negative.

European Call Spread. Let us test our algorithm in the case of a Call Spread option with payoff $(X_T - K_1)^+ - (X_T - K_2)^+$. We pick $K_1 = 90$ and $K_2 = 110$. The true price (PDE) is $C_{PDE} = 11.20$ and the Black-Scholes price with the mid-volatility 15% is $C_{BS} = 9.52$.

In this case, the Monte Carlo approach can capture the right magnitude of the price. From the following figure, we see that the choice of basis functions (here polynomial with different maximal degree) and time step clearly affects the price estimate. However, as the estimator is low-biased, one possibility is to use the pricing algorithm with different parameters and take the maximum of simulated prices.

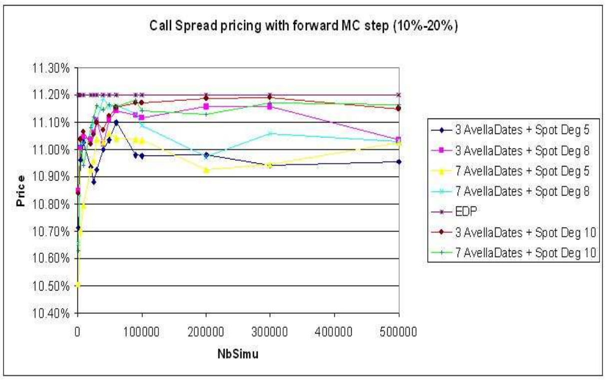

Figure 6.5: Call Spread pricing with Forward step for different backward dates and polynomial basis functions

Also the choice for the backward step diffusion coefficient $\hat{\sigma}$ can influence numerical results. In fact, theoretically this scheme does not depend on $\hat{\sigma}$, but since we have to use Monte Carlo regression, then the scheme is very sensible to this parameter. To fix this issue, one possible way is to price with different values for $\hat{\sigma}$. It is reported in Guyon and Henry-Labordère et al. [47] that, among all the constant volatilities tested, the best result for European Call Spread is obtained with the mid-volatility $\hat{\sigma} = 15\%$. But we need to emphasize that the mid-volatility may not be the optimal choice for others options.

Asian Call. We begin our tests for path-dependent options with an Asian Call. The payoff is $(A_T - K)^+$ where $A_T = \frac{T}{12}\sum_{i=1}^{12} X_{t_i}$ the monthly sampled arithmetic average. The PDE price is $C_{PDE} = 4.85$ with $K = 100$, $\bar{\sigma} = 20\%$, $\underline{\sigma} = 10\%, 15\%, 20\%$. As explained in the section on path-dependent options, we try two algorithms for Asian options in the numerical tests. The first one is a Finite Difference inspired Monte Carlo scheme (denoted by FC+MC) and the second one is a purely Monte Carlo scheme. Both algorithms give good results with suitable parameters. But the first one is time and memory-consuming, so we can't compute prices with more than 100000 simulations in the case of 250 subdivisions.

And for the second algorithm, we can construct basis functions either with only Spot price or with both Spot price and average. The simulated prices in these two cases have good precision with a large number of simulations. When constructing only with Spot price, we make an approximation that

$$\mathbb{E}\left[\cdot|X_{t_{i-1}}, A_{t_{i-1}}\right] \approx \mathbb{E}\left[\cdot|X_{t_{i-1}}\right].$$

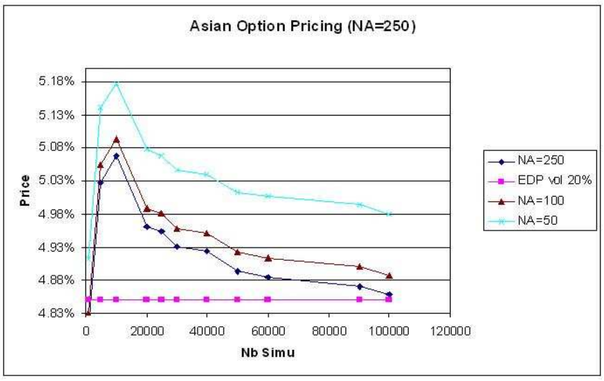

Figure 6.6: Asian Call Pricing with FD+MC Vol min=15% Vol max=20% for different subdivisions of the average (NA)

Figure 6.7: Asian Call Pricing with MC Backward step only Vol min=15% Vol max=20%

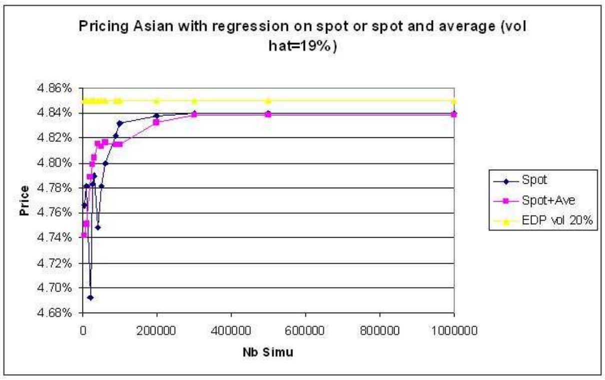

Figure 6.8: Asian Call Pricing with MC Forward step Vol min=10% Vol max=20% Vol diffusion $\hat{\sigma}$=19%

We have computed the price of the above Asian Call option with different values for the backward step diffusion volatility $\hat{\sigma}$. It turns out the best result is obtained with $\hat{\sigma} = 19\%$ (which is close to the Max Vol $\bar{\sigma} = 20\%$).

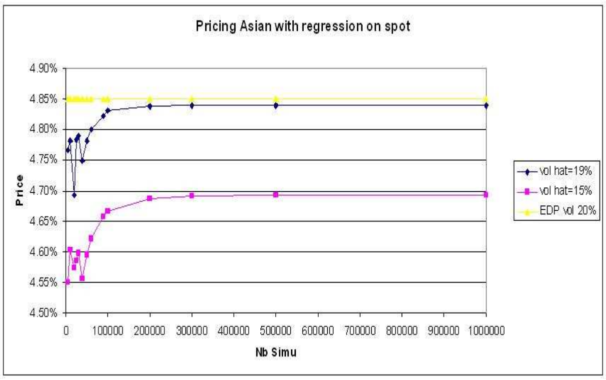

Figure 6.9: Asian Call Pricing with MC Forward step Vol min=10% Vol max=20% for different diffusion vols $\hat{\sigma}$

Asian Call Spread. As European Call Spread for non-path-dependent options, Asian Call Spread is more interesting to test our algorithm for path-dependent options. The payoff is $(A_T - K_1)^+ - (A_T - K_2)^+$ where $A_T = \frac{T}{12}\sum_{i=1}^{12} X_{t_i}$ the monthly sampled arithmetic average and $K_1 < K_2$. For our numerical tests,we take $K_1 = 90$, $K_2 = 110$. The PDE price $C_{PDE} = 10.67$ with $\bar{\sigma} = 20\%$, $\underline{\sigma} = 10\%$, $C_{PDE} = 10.16$ with $\bar{\sigma} = 20\%$, $\underline{\sigma} = 15\%$, and $C_{PDE} = 9.85$ with $\bar{\sigma} = \underline{\sigma} = 20\%$. From our tests, we notice that by using the value of an European Call option with same strike as control variate, the prices have better precision than the ones obtained without control variate. Unlike for European options, it is necessary to apply control variates for path-dependent options.

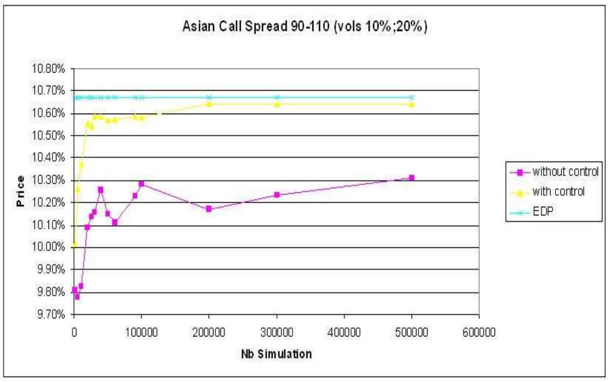

Figure 6.10: Asian Call Spread Pricing with control variate Vol min=10% Vol max=20%

Call Sharpe. To finish our numerical tests, let us test the algorithms with a Call Sharpe option paying $(X_T - 100)^+/\sqrt{V_T}$ where $V_T = \frac{1}{T}\sum_{l=1}^{12}\left(ln\frac{X_{t_l}}{X_{t_{l-1}}}\right)^2$ is the realized volatility computed using monthly returns.

As mentioned in the section on path-dependent options pricing, it is notably difficult to find a convenient basis to compute the conditional expectations and we assume as an approximation that

$$\mathbb{E}\left[\cdot|X_{t_{i-1}}, A^1_{t_{i-1}}, A^2_{t_{i-1}}\right] \approx \mathbb{E}\left[\cdot|X_{t_{i-1}}\right].$$

We notice also that with well chosen control variates, the estimated prices have better precision.

Vol min max	PDE	without control	control for Y	control for Y,Z	control for Y,Z(HC)
15%-15%	40.71	41.09	40.65	40.65	40.65
15%-20%	47.50	45.53	46.36	46.90	47.09
10%-20%	57.66	51.64	54.1	54.05	56.04

Table 6.1: Calls Sharpe pricing with Backward step only for different Vols Min and Vol Max

European Call with 2 underlying assets. We also test the algorithm in the case of an European Call option with 2 underlying assets. The payoff is $(\frac{X_T^1+X_T^2}{2} - K)^+$. We take $K = 100$. The PDE price is $C_{PDE} = 5.98$ when the correlation $\rho = 100\%$ and $C_{PDE} = 5.58$ when $\rho = 70\%$. As showed on following figures, the algorithm with backward step produces a good estimation of the price when there is no uncertainty on both volatilities and correlations. But in the case of uncertain volatilities or correlations, we need to improve the estimated prices precision by using suitable backward step diffusion volatilities and correlations or more appropriate basis functions.

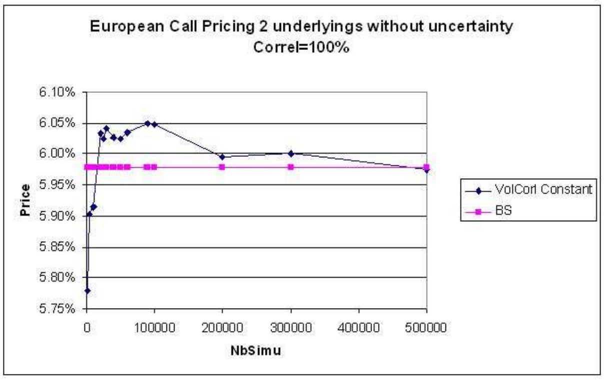

Figure 6.11: Call (100%) Pricing 2 underlyings Vol1 min=Vol1 max=20% Vol2 min=Vol2 max=10% Correl=100%

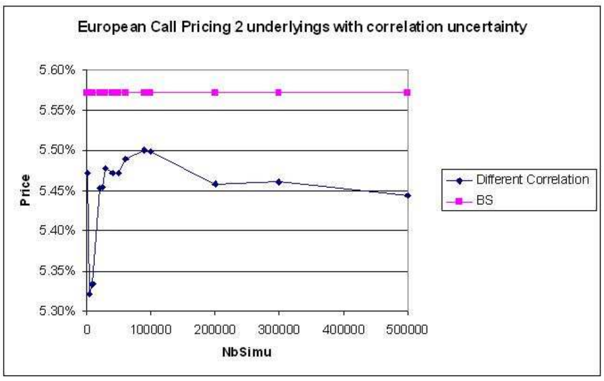

Figure 6.12: Call (100%) Pricing 2 underlyings Vol1 min=Vol1 max=20% Vol2 min=Vol2 max=10% Correl Min=50% Correl Max=70%

6.8 An algorithm without regression

We see that in our previous tests, using the Monte Carlo regression method to approximate conditional expectation often leads to time consuming pricing procedure or mis-pricing. In this section, we present an alternative way to implement the Scheme Guyon and Henry-Labordère. We derive a new algorithm where no Monte Carlo simulations are needed. According to the paper of Carr and Madan [19], we know that under some assumptions, the price of an European style option can be presented like a continuous sum of puts and calls (with BS model) with different strikes. There are closed formulas for conditional expectations of these puts and calls. So if we write price Y at every backward date on functions of puts and calls with BS model, then we can use closed formulas to approximate conditional expectations in the formulas for option price and Gamma with UVM. By doing so, there is no Monte Carlo regression to perform, so the program is quicker to execute and there is no simulation error. But we also need to choose a diffusion volatility to calculate BS puts and calls prices and Gammas like the previous method. And the quality of the approximation by payoffs of puts and calls is very important to have the right price. And for path-dependent options, we need to integrate the path-dependent variables in the puts and calls basis. In the following, to simplify the formulas, we denote the Malliavin weight for Gamma Γ by C..

In one-dimensional case, the formulas (6.3.4) become:

$$\begin{cases} X_{t_i} = X_0 e^{-\hat{\sigma}^2 \frac{t_i}{2} + \hat{\sigma} W_{t_i}} \\ Y_{t_{i-1}} = \mathbb{E}\left[Y_{t_i} | X_{t_{i-1}}\right] + \left(f\left(X_{t_{i-1}}, \Gamma_{t_{i-1}}\right) - \frac{1}{2}\hat{\sigma}^2 X_{t_{i-1}}^2 \Gamma_{t_{i-1}}\right)\Delta t_i \\ \Delta t_i \hat{\sigma} X_{t_{i-1}}^2 \Gamma_{t_{i-1}} = \mathbb{E}\left[Y_{t_i} \cdot \left(\frac{\Delta W_{t_i}^2}{\hat{\sigma}\Delta t_i} - \Delta W_{t_i} - \frac{1}{\hat{\sigma}}\right) | X_{t_{i-1}}\right] \end{cases} \tag{6.8.1}$$

We have closed formulas for $Y^{BS,i}$ and $\Gamma^{BS,i}$. If we write

$$\begin{aligned} Y_{t_N=T} &= \sum_{i=1}^{I} \alpha_i (X_T - K_i)^+ + \sum_{j=1}^{J} \beta_j (K_j - X_T)^+ \\ &= \sum_{i=1}^{I} \alpha_i Y_T^{BS,Call,T,i} + \sum_{j=1}^{J} \beta_j Y_T^{BS,Put,T,j} \end{aligned}$$

Then, we get

$$\begin{aligned} \Gamma_{t_{N-1}} &= \mathbb{E}\left[C_{t_N} Y_{t_N} | \mathcal{F}_{t_{N-1}}\right] \\ &= \sum_{i=1}^{I} \alpha_i \mathbb{E}\left[C_{t_N} Y_{t_N}^{BS,Call,t_N,i} | \mathcal{F}_{t_{N-1}}\right] + \sum_{j=1}^{J} \beta_j \mathbb{E}\left[C_{t_N} Y_{t_N}^{BS,Put,t_N,j} | \mathcal{F}_{t_{N-1}}\right] \\ &= \sum_{i=1}^{I} \alpha_i \Gamma_{t_{N-1}}^{BS,Call,t_N,i} + \sum_{j=1}^{J} \beta_j \Gamma_{t_{N-1}}^{BS,Put,t_N,j} \end{aligned}$$

And

$$\begin{aligned}
Y_{t_{N-1}} &= \mathbb{E}\left[Y_{t_N=T}|\mathcal{F}_{t_{N-1}}\right] + \left(\frac{1}{2}\left((\sigma^*)^2 - \hat{\sigma}^2\right) X^2_{t_{N-1}}\Gamma_{t_{N-1}}\right)\Delta t_N \\
&= \sum_{i=1}^{I}\alpha_i\left[Y^{BS,Call,t_N,i}_{t_{N-1}} + \left(\frac{1}{2}\left((\sigma^*)^2 - \sigma^2\right) X^2_{t_{N-1}}\Gamma^{BS,Call,,t_N,i}_{t_{N-1}}\right)\Delta t_N\right] \\
&+ \sum_{j=1}^{J}\beta_j\left[Y^{BS,Put,t_N,j}_{t_{N-1}} + \left(\frac{1}{2}\left((\sigma^*)^2 - \sigma^2\right) X^2_{t_{N-1}}\Gamma^{BS,Put,t_N,j}_{t_{N-1}}\right)\Delta t_N\right]
\end{aligned}$$

Then on t_{N-1}, we can find new α_i and β_j such that

$$\begin{aligned}
Y_{t_{N-1}} &\cong \sum_{i=1}^{I}\alpha_i(S_{t_{N-1}} - K_i)^+ + \sum_{j=1}^{J}\beta_j(K_j - S_{t_{N-1}})^+ \\
&= \sum_{i=1}^{I}\alpha_i Y^{BS,Call,t_{N-1},i}_{t_{N-1}} + \sum_{j=1}^{J}\beta_j Y^{BS,Put,t_{N-1},j}_{t_{N-1}}
\end{aligned}$$

So

$$\begin{aligned}
\Gamma_{t_{N-2}} &= \mathbb{E}\left[C_{t_{N-1}}Y_{t_{N-1}}\Delta W^2_{t_{N-1}}|\mathcal{F}_{t_{N-2}}\right] \\
&= \sum_{i=1}^{I}\alpha_i\mathbb{E}\left[C_{t_{N-1}}Y^{BS,Call,t_{N-1},i}_{t_{N-1}}|\mathcal{F}_{t_{N-2}}\right] + \sum_{j=1}^{J}\beta_j\mathbb{E}\left[C_{t_{N-1}}Y^{BS,Put,t_{N-1},j}_{t_{N-1}}|\mathcal{F}_{t_{N-2}}\right] \\
&= \sum_{i=1}^{I}\alpha_i\Gamma^{BS,Call,t_{N-1},i}_{t_{N-2}} + \sum_{j=1}^{J}\beta_j\Gamma^{BS,Put,t_{N-1},j}_{t_{N-2}}
\end{aligned}$$

$\vdots$

By continuing this routine, we can get Y_0 which is the price at the initial pricing date.

In this method, the question is how to choose the different Strikes and how to obtain the coefficients before Calls and Puts. One possible way is to use the same grid for Spot Prices and Strikes. And we choose a particular strike which generally could be the initial spot price(or a price close to this one), Puts are with strikes smaller than this particular strike and Calls are with strikes bigger than this one. Then by identifying the option price Y and the sum of Calls and Puts for each Spot Price in price grid, we get two linear systems to solve. Thanks to the particular choice of Calls and Puts, these two systems are an upper triangular one and a lower triangular one which are easy to solve numerically. In this method, two parameters are important. The first one is number of backward dates who represents the discretization of the time dimension for a continuous process. And the second one is number of subdivision for spot price dimension. In order to better approach a continuous process by a discrete version, we need more backward dates, but more backward dates mean more approximations of option price by sum of Calls and Puts payoff, then probably more errors. So we should choose this parameter cautiously. In order to approximate better option price by sum of Calls and Puts, it's natural to subdivide more the Spot Price dimension thus have more Calls and Puts with strikes closer to each other. But more subdivision means more computational time and particularly this may introduce some instability. As for the time dimension, this parameter should be chosen carefully.

We apply this new algorithm to both path- and non-path-dependent options. As we can see from the tests, with 30-50 backward dates and 300-500 Spot Price subdivision, we can have a price estimation with good precision. We use a non-uniform subdivision with concentration around the initial Spot for Spot price. Idea behind this is to better capture the convexity around the initial spot.

But this approach is very similar to the Finite Difference method: subdivide each dimension, then compute the price by a roll-back process. Particularly in the cases with path-dependent variables or of multi-dimension, this approach suffers the same problem as Finite Difference method. So in thess cases we prefer the Monte Carlo method.

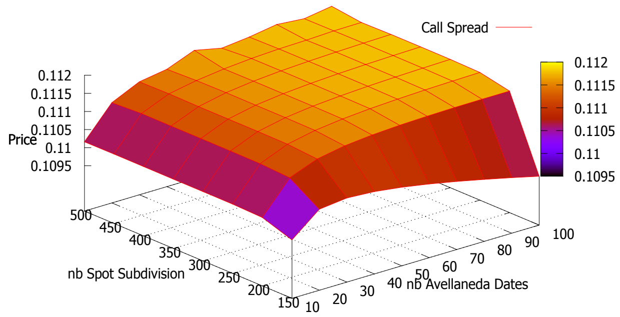

Figure 6.13: Call Spread (90%-110%) Pricing without MC regression Vol min=10% Vol max=20%

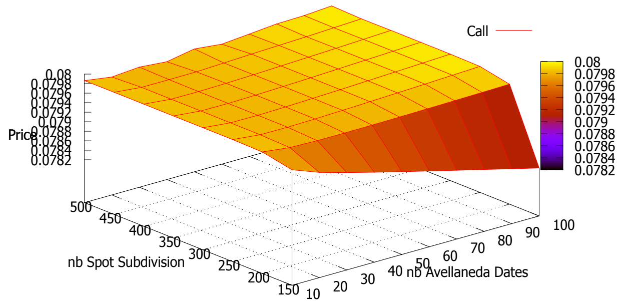

Figure 6.14: Call (100%) Pricing without MC regression Vol min=10% Vol max=20%

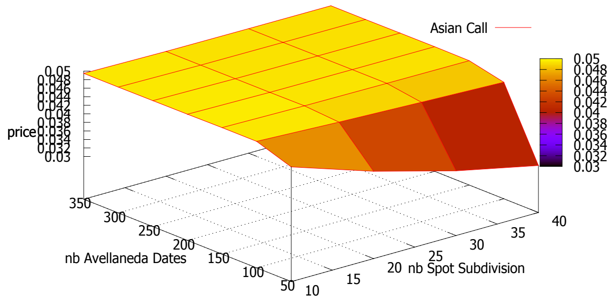

Figure 6.15: Asian Call (100%) Pricing without MC regression Vol min=10% Vol max=20%

6.9 Conclusion

From our numerical tests (see Appendix 6.10 for more results), we generally observe that the Monte Carlo method performs well for non-path-dependent options and can provide good precision prices for path-dependent ones with well chosen basis functions.

In order to get more precise results with this method, we should improve the approximation of conditional expectations by using better regression procedure, local support basis functions, suitable control variates and non-parametric regressions in higher dimension.

6.10 Appendix

Here we report more results of our numerical tests.

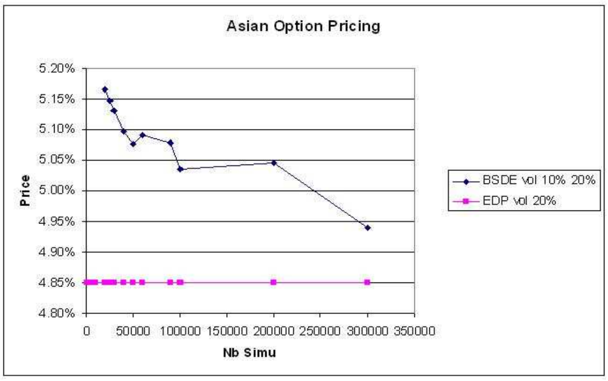

Figure 6.16: Asian Call Pricing with FD+MC Vol min=10% Vol max=20%

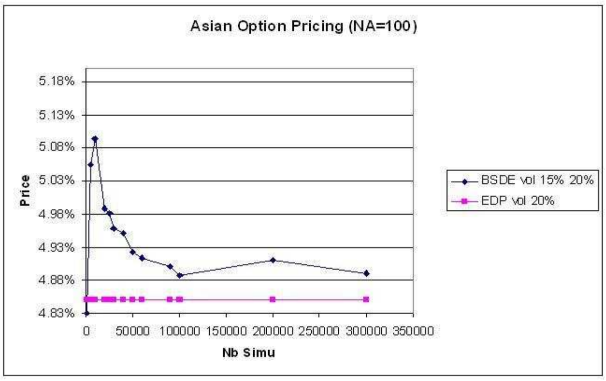

Figure 6.17: Asian Call Pricing with FD+MC Vol min=15% Vol max=20%

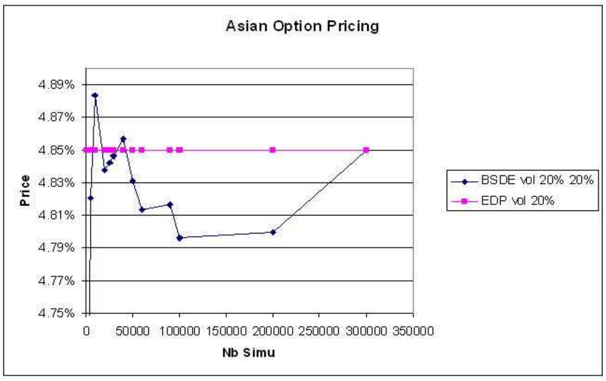

Figure 6.18: Asian Call Pricing with FD+MC Vol min=20% Vol max=20%

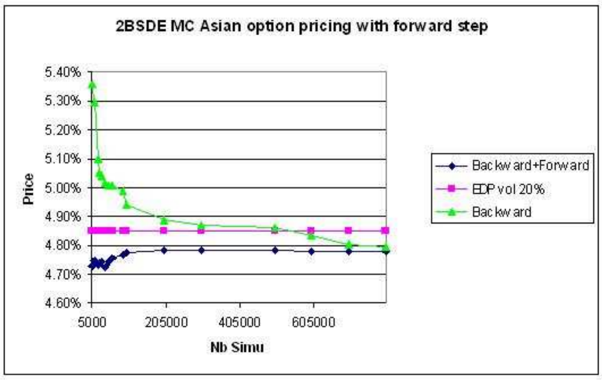

Figure 6.19: Asian Call Pricing with MC Backward step and Forward step Vol min=10% Vol max=20%

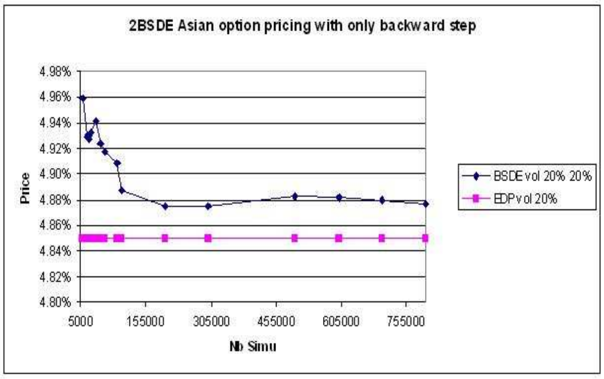

Figure 6.20: Asian Call Pricing with MC Backward step only Vol min=20% Vol max=20%

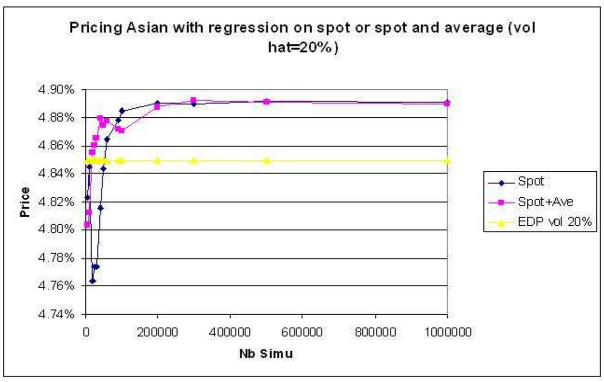

Figure 6.21: Asian Call Pricing with MC Forward step Vol min=10% Vol max=20% Vol diffusion $\hat{\sigma}$=20%

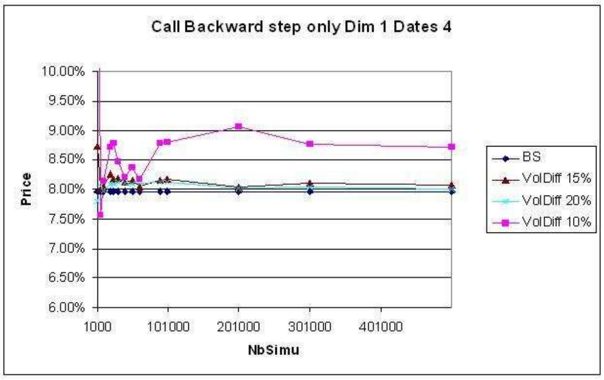

Figure 6.22: European Call Pricing with Backward step only Vol min=10% Vol max=20% for different diffusion vols $\hat{\sigma}$

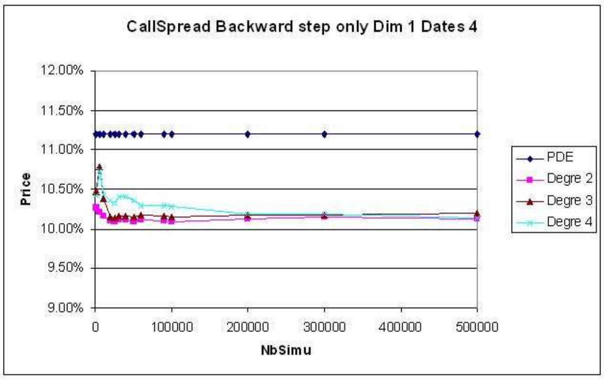

Figure 6.23: European Call Spread Pricing with Backward step only Vol min=10% Vol max=20% for different polynomial basis

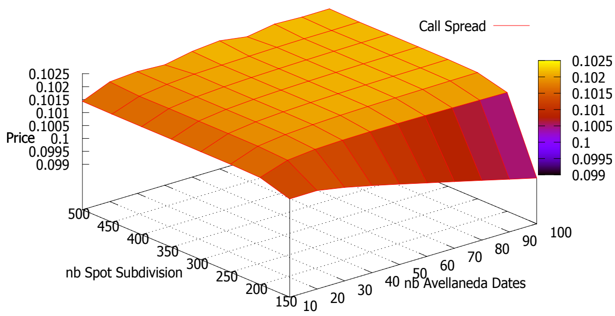

Figure 6.24: Call Spread (90%-110%) Pricing without MC regression Vol min=15% Vol max=20%

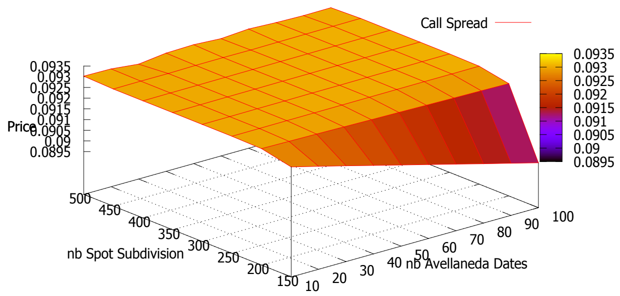

Figure 6.25: Call Spread (90%-110%) Pricing without MC regression Vol min=Vol max=20%

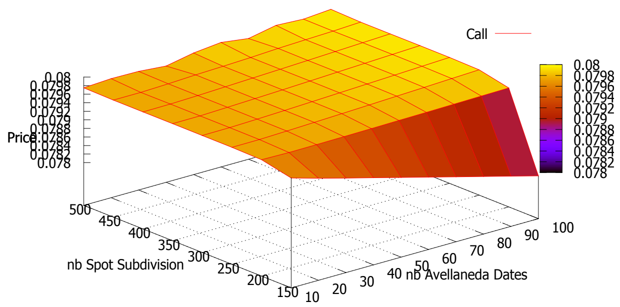

Figure 6.26: Call (100%) Pricing without MC regression Vol min=15% Vol max=20%

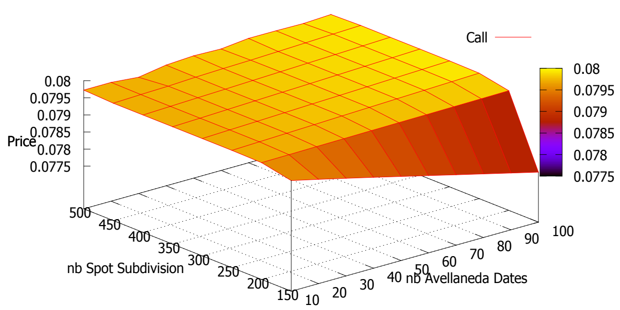

Figure 6.27: Call (100%) Pricing without MC regression Vol min=Vol max=20%

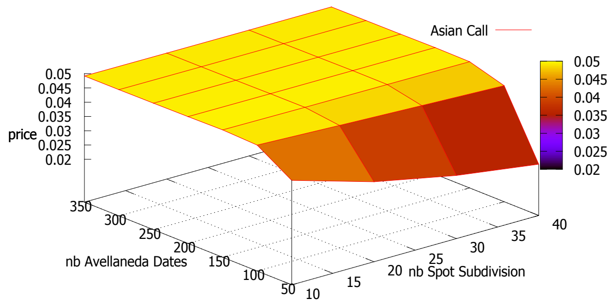

Figure 6.28: Asian Call (100%) Pricing without MC regression Vol min=15% Vol max=20%

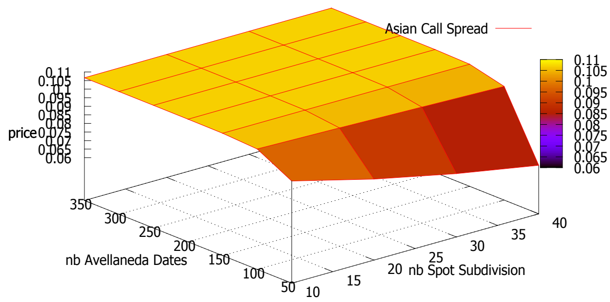

Figure 6.29: Asian Call Spread (90%-110%) Pricing without MC regression Vol min=10% Vol max=20%

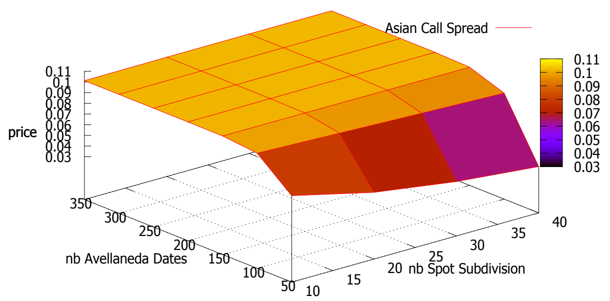

Figure 6.30: Asian Call Spread (90%-110%) Pricing without MC regression Vol min=15% Vol max=20%

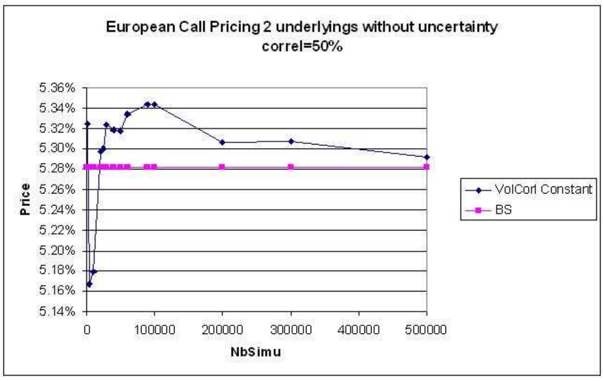

Figure 6.31: Call (100%) Pricing 2 underlyings Vol1 min=Vol1 max=20% Vol2 min=Vol2 max=10% Correl=50%

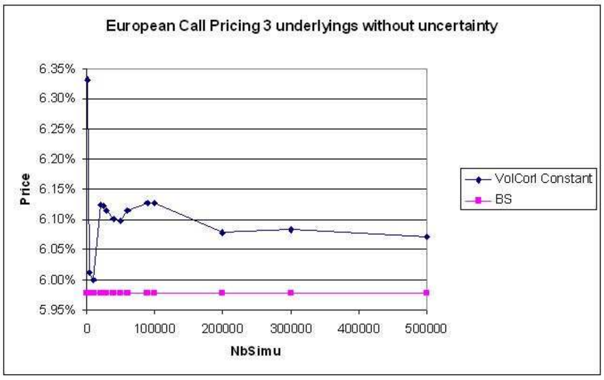

Figure 6.32: Call (100%) Pricing 3 underlyings Vol1 min=Vol1 max=20% Vol2 min=Vol2 max=15% Vol2 min=Vol2 max=10% Correl=100%

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